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The spectrum of Meta($K_3 + e > P_4, \lambda$) and Meta($K_3 + e > H_4, \lambda$) with any λ ☆

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Abstract

Let $(X, \mathcal{B})$ be a $(\lambda K_v, G_1)$-design and G_2 a subgraph of G_1. Define sets $\mathcal{B}(G_2)$ and $\mathcal{D}(G_1 \setminus G_2)$ as follows: for each block $B \in \mathcal{B}$, partition B into copies of G_2 and $G_1 \setminus G_2$ and place the copy of G_2 in $\mathcal{B}(G_2)$ and the edges belonging to the copy of $G_1 \setminus G_2$ in $\mathcal{D}(G_1 \setminus G_2)$. If the edges belonging to $\mathcal{D}(G_1 \setminus G_2)$ can be assembled into a collection $\mathcal{D}(G_2)$ of copies of G_2, then $(X, \mathcal{B}(G_2) \cup \mathcal{D}(G_2))$ is a $(\lambda K_v, G_2)$-design, called a *metamorphosis* of the $(\lambda K_v, G_1)$-design $(X, \mathcal{B})$. For brevity we denote such $(\lambda K_v, G_1)$-design $(X, \mathcal{B})$ with a metamorphosis into $(\lambda K_v, G_2)$-design $(X, \mathcal{B}(G_2) \cup \mathcal{D}(G_2))$ by $(\lambda K_v, G_1 > G_2)$-design. Let Meta($G_1 > G_2, \lambda$) denote the set of all integers v such that there exists a $(\lambda K_v, G_1 > G_2)$-design. In this paper we completely determine the set Meta($K_3 + e > P_4, \lambda$) or Meta($K_3 + e > H_4, \lambda$) when the admissible conditions are satisfied, for any λ.

Keywords: G-design; Metamorphosis; Balanced design; Differences; Cyclic and 1-rotational design

1. Introduction

Let G and K be simple finite graphs, and let λK denote the graph K with each of its edges replicated λ times. A λ*-fold* G*-design* of order v ($(\lambda K_v, G)$-design in short) is a pair $(X, \mathcal{B})$, where X is the vertex set of λK_v and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G whose edges partition the edges of λK_v. If $\lambda = 1$, we drop the term "1-fold". For terms not defined in this paper or results not explicitly cited the reader is referred to *The CRC Handbook of Combinatorial Designs* [7].

Let $(X, \mathcal{B})$ be a $(\lambda K_v, G_1)$-design and G_2 a subgraph of G_1. Define sets $\mathcal{B}(G_2)$ and $\mathcal{D}(G_1 \setminus G_2)$ as follows: for each block $B \in \mathcal{B}$, partition B into copies of G_2 and $G_1 \setminus G_2$ and place the copy of G_2 in $\mathcal{B}(G_2)$ and the edges belonging to the copy of $G_1 \setminus G_2$ in $\mathcal{D}(G_1 \setminus G_2)$. If the edges belonging to $\mathcal{D}(G_1 \setminus G_2)$ can be assembled into a collection $\mathcal{D}(G_2)$ of copies of G_2, then $(X, \mathcal{B}(G_2) \cup \mathcal{D}(G_2))$ is a $(\lambda K_v, G_2)$-design, called a *metamorphosis* of the λ-fold G_1-design $(X, \mathcal{B})$.

For brevity we denote such G_1-design of $\lambda K_v(X, \mathcal{B})$ with a metamorphosis into G_2-design of $\lambda K_v(X, \mathcal{B}(G_2) \cup \mathcal{D}(G_2))$ by $(\lambda K_v, G_1 > G_2)$-design, or $(X, \mathcal{B}, \mathcal{B}(G_2) \cup \mathcal{D}(G_2))$. Let Meta($G_1 > G_2, \lambda$) denote the *spectrum* for $(\lambda K_v, G_1 > G_2)$-designs, i.e. the set of all integers v such that there exists a $(\lambda K_v, G_1 > G_2)$-design.

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Recently the spectrum of Meta($G_1 > G_2, \lambda$) has been determined for each pair $(G_1, G_2) = (K_{3,3}, C_6)$, (4-wheel, bowtie), (4-wheel, C_4), (K_4, K_3), (K_4, K_3+e), (K_4, C_4), (K_4+e, K_4), and (K_4-e, K_3+e) with any λ by several researchers (see [1–3,5,6,10,12,14]). For a brief history of further work on the metamorphosis of G-design, see the introduction to [2]. Other recent papers on metamorphosis are included in [11,13,15,16].

In what follows we will denote the copy of K_3+e (kite) with vertices a, b, c, d containing the K_3 (a, b, c) and the dangling edge cd by $(a, b, c) - d$, and the copy of P_4 with vertices a, b, c, d containing the edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$ by (a, b, c, d). Suppose that $\mathcal{B}$ is a collection of isomorphic copies of the graph K_3+e. Define $\mathcal{B}^*$ and $\mathcal{D}_{\mathcal{B}}$ as follows: For each block $(a, b, c) - d \in \mathcal{B}$, delete the edge $\{b, c\}$ and place the copy of P_4 in $\mathcal{B}^*$ and the deleted edge in $\mathcal{D}_{\mathcal{B}}$. Further, let $(X, \mathcal{B})$ be a $(\lambda K_v, K_3+e)$-design, then $(X, \mathcal{B}, \mathcal{B}^* \cup \mathcal{B}')$ is a $(\lambda K_v, K_3+e > P_4)$-design if $\mathcal{D}_{\mathcal{B}}$ can be partitioned into a collection $\mathcal{B}'$ of copies of P_4. In this paper, we deal with Meta($K_3+e > P_4, \lambda$) for any λ.

Necessary conditions: Recall that a $(\lambda K_v, K_3+e)$-design exists if and only if $\lambda v(v-1) \equiv 0 \pmod 8$ (see [10]); a $(\lambda K_v, P_4)$-design exists if and only if $\lambda v(v-1) \equiv 0 \pmod 6$ (see [9]). The following are the necessary conditions for their existence:

$(\lambda K_v, K_3+e)$-design		$(\lambda K_v, P_4)$-design	
$\lambda \pmod 4 > 0$	v	$\lambda \pmod 3 > 0$	v
1, 3	$v \equiv 0, 1 \pmod 8$	1, 2	$v \equiv 0, 1 \pmod 3$
2	$v \equiv 0, 1 \pmod 4$	3	all $v \geq 4$
4	all $v \geq 4$		

We require the intersection of those conditions for the possible existence of a $(\lambda K_v, K_3+e > P_4)$-design, which are as listed below.

$\lambda \pmod{12} > 0$	v
1, 5, 7, 11	$v \equiv 0, 1, 9, 16 \pmod{24}$
2, 10	$v \equiv 0, 1, 4, 9 \pmod{12}$
3, 9	$v \equiv 0, 1 \pmod 8$
4, 8	$v \equiv 0, 1 \pmod 3$
6	$v \equiv 0, 1 \pmod 4$
12	$v \geq 4$

Consequently in the subsequent sections we will deal with the set Meta($K_3+e > P_4, \lambda$) with the cases λ equal to 1, 2, 3, 4, 6, 12; to solve these cases we use the *difference method* (see [4,8]). Note that, throughout the paper, applying the difference method gives cyclic or 1-rotational designs.

Let $G = (V(G), E(G))$ be a graph with $V(G) \subseteq Z_n$ and $G + i = \{\{a+i, b+i\} | \{a, b\} \in E(G)\}$, $i \in Z_n$. A $(\lambda K_v, G)$-design $(X, \mathcal{B})$ is *cyclic* or 1-*rotational* if $X = Z_v$ or $X = Z_{v-1} \cup \{\infty\}$, respectively, and $B + 1 \in \mathcal{B}$ whenever $B \in \mathcal{B}$ (where $\infty + 1 = \infty$ is understood).

Let $D_n = \{d \in Z_n : 1 \leq d \leq \lfloor \frac{n}{2} \rfloor\}$. The elements of D_n are called *differences* of Z_n. Let G be a graph with $V(G) \subseteq Z_n \cup \{\infty\}$. The *list of differences* of G is the multiset

$$\triangle G = \{a - b \mid a, b \in V(G) - \{\infty\}, \{a, b\} \in E(G)\}.$$

If $\mathcal{B}$ is a collection of copies of G with vertices in $Z_n \cup \{\infty\}$, the list of differences of $\mathcal{B}$ is the multiset $\triangle \mathcal{B} = \bigcup_{B \in \mathcal{B}} \triangle B$.

If the vertices of G are in $Z_n \cup \{\infty\}$, the *orbit* of G under Z_n is the set $\{G + i : i \in Z_n\}$. To describe a cyclic or a 1-rotational $(\lambda K_v, G)$-design, it is sufficient to give a collection of *base blocks*, i.e. a system of representatives for its orbits under Z_v or Z_{v-1}. Let $\mathcal{B}$ be a collection of copies of G with vertices in X. It is easy to see that:

1. if $X = Z_v$ and $\triangle \mathcal{B} = \lambda D_v$, then the union of the orbits of the graphs of $\mathcal{B}$ under Z_v is a cyclic $(\lambda K_v, G)$-design;
2. if $X = Z_{v-1} \cup \{\infty\}$, $\triangle \mathcal{B} = \lambda D_{v-1}$, and $\sum_{B \in \mathcal{B}} d_B(\infty) = \lambda$, where $d_B(\infty)$ is the degree of ∞ in B, then the union of the orbits of the graphs of $\mathcal{B}$ under Z_{v-1} is a 1-rotational $(\lambda K_v, G)$-design.

A $(\lambda K_v, G)$-design is said to be *balanced* if each vertex belongs to exactly r blocks. A cyclic $(\lambda K_v, G)$-design is a balanced $(\lambda K_v, G)$-design. Denote a balanced $(\lambda K_v, P_4)$-design by $(\lambda K_v, H_4)$-design. In [9] it is proved that