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SPG-reguli, SPG-systems, BLT-sets and sets with the BLT-property

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Abstract

This paper is a survey on SPG-reguli, SPG-systems, BLT-sets and sets with the BLT-property. It is shown how from these sets generalized quadrangles, partial geometries and semi-partial geometries can be constructed. Many examples are given and open problems are stated. There are also some new results.

Keywords: SPG-regulus; SPG-system; BLT-set; BLT-property; Generalized quadrangle; Partial geometry; Semi-partial geometry

1. Introduction

This paper is a survey on SPG-reguli, SPG-systems, BLT-sets and sets with the BLT-property. These structures, which are all constructed in Galois spaces, were introduced roughly between 1980 and 2000. Also particular m-systems will be discussed. Apart from being interesting in their own, all these objects are used to construct new point-line incidence geometries, such as partial geometries, semi-partial geometries and generalized quadrangles; also new strongly regular graphs arise. Further, we will see that these structures are closely related. It is our purpose to give the necessary definitions, the main properties, applications (in particular the construction of interesting point-line geometries), and open problems. Besides the relevant "old" results, also new ones will be stated.

2. SPG-reguli

2.1. SPG-reguli

SPG-reguli were introduced by Thas [37].

An SPG-*regulus* is a set $\mathcal{R}$ of m-dimensional subspaces $\mathrm{PG}^{(1)}(m,q)$, $\mathrm{PG}^{(2)}(m,q), \ldots, \mathrm{PG}^{(r)}(m,q)$, $r > 1$, of $\mathrm{PG}(n,q)$, $n > 1$, satisfying the following conditions.

(i) $\mathrm{PG}^{(i)}(m,q) \cap \mathrm{PG}^{(j)}(m,q) = \emptyset$ for all $i \neq j$.
(ii) If $\mathrm{PG}(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $\mathcal{R} - \{\mathrm{PG}^{(i)}(m,q)\}$. If $\mathrm{PG}(m+1,q)$ has no point in common with $\mathrm{PG}^{(j)}(m,q)$ for all $j \neq i$, then it is called a *tangent* $(m+1)$-*space* or a *tangent space* of $\mathcal{R}$ at $\mathrm{PG}^{(i)}(m,q)$.

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(iii) If the point x of $\mathrm{PG}(n,q)$ is not contained in an element of $\mathcal{R}$, then it is contained in a constant number θ ($\theta \geq 0$) of tangent spaces of $\mathcal{R}$.

By (i) we have $n \geq 2m+1$, and if $n = 2m+1$ then there are no tangent spaces (and so $\alpha = r-1$).

In [37] it is shown that

$$\alpha(q-1) \text{ divides } (r-1)(q^{m+1}-1)$$

and

$$\theta = \frac{(\alpha(q^{n-m}-1)-(r-1)(q^{m+1}-1))rq^{m+1}}{\alpha((q^{n+1}-1)-r(q^{m+1}-1))}.$$

Hence $\theta = 0$ if and only if $\alpha(q^{n-m}-1) = (r-1)(q^{m+1}-1)$.

2.2. *Semi-partial geometries*

A *semi-partial geometry* (SPG) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ of points and lines, with $P \neq \emptyset \neq B$, satisfying the following axioms.

(i) Each point is incident with $1+t$ ($t \geq 1$) lines and two distinct points are incident with at most one line.
(ii) Each line is incident with $1+s$ ($s \geq 1$) points and two distinct lines are incident with at most one point.
(iii) If two points are not collinear, then there are μ ($\mu > 0$) points collinear with both.
(iv) If a point x and a line L are not incident, then there are either 0 or α ($\alpha \geq 1$) points which are collinear with x and incident with L (i.e., there are either 0 or α points x_i and either 0 or α lines L_i respectively, such that $x\mathrm{I}L_i\mathrm{I}x_i\mathrm{I}L$).

The parameter α is called the *incidence number* of $\mathcal{S}$ and the semi-partial geometry $\mathcal{S}$ is denoted by $\mathrm{spg}(s,t,\alpha,\mu)$. Semi-partial geometries were introduced by Debroey and Thas [9].

An SPG with $\alpha = 1$ is called a *partial quadrangle* (PQ). Partial quadrangles were introduced and studied by Cameron [7]. If in (iv) there are always α points collinear with x and incident with L, then the $\mathrm{spg}(s,t,\alpha,\mu)$ is called a *partial geometry* (PG). In such a case (iii) is automatically satisfied. A semi-partial geometry which is not a partial geometry is called a *proper* semi-partial geometry. Partial geometries were introduced by Bose [3]. A partial geometry which is also a partial quadrangle is called a *generalized quadrangle* (GQ); generalized quadrangles were introduced by Tits [44].

An SPG is a PG if and only if $\mu = (t+1)\alpha$. If $\mu \neq (t+1)\alpha$, then $t \geq s$ for any $\mathrm{spg}(s,t,\alpha,\mu)$; see [9]. Further, the dual of an SPG $\mathcal{S}$ is again an SPG if and only if either $s = t$ or $\mathcal{S}$ is a PG; see [9]. Finally we remark that the point graph of an $\mathrm{spg}(s,t,\alpha,\mu)$ is a strongly regular graph

$$\mathrm{srg}\left(1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, s(t+1), s-1+t(\alpha-1), \mu\right).$$

For an excellent survey on semi-partial geometries we refer to [10].

2.3. *Semi-partial geometries arising from SPG-reguli*

Let $\mathcal{R}$ be an SPG-regulus, i.e., let $\mathcal{R}$ be a set of m-dimensional subspaces $\mathrm{PG}^{(1)}(m,q), \mathrm{PG}^{(2)}(m,q), \ldots, \mathrm{PG}^{(r)}(m,q)$, $r > 1$, of $\mathrm{PG}(n,q)$ satisfying (i)–(iii) of 2.1. Now we embed $\mathrm{PG}(n,q)$ as a hyperplane in $\mathrm{PG}(n+1,q)$, and we define an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ of points and lines as follows:

- POINTS of $\mathcal{S}$ are the points in $\mathrm{PG}(n+1,q) - \mathrm{PG}(n,q)$;
- LINES of $\mathcal{S}$ are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of $\mathcal{R}$ but are not contained in $\mathrm{PG}(n,q)$;
- INCIDENCE is that of $\mathrm{PG}(n+1,q)$.

Then $\mathcal{S}$ is an $\mathrm{spg}(q^{m+1}-1, r-1, \alpha, (r-\theta)\alpha)$.

The SPG is a PG if and only if $\theta = 0$; if $\mathcal{S}$ is not a PG, i.e., if $\theta \neq 0$, or, equivalently, $\alpha(q^{n-m}-1) \neq (r-1)(q^{m+1}-1)$, then $t \geq s$ implies $r \geq q^{m+1}$.