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Hexagon kite systems☆

Lucia Gionfriddo

Dipartimento di Matematica e Informatica, Università di Catania, Viale A.Doria n.6, 95125 Catania, Italy

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1. Introduction

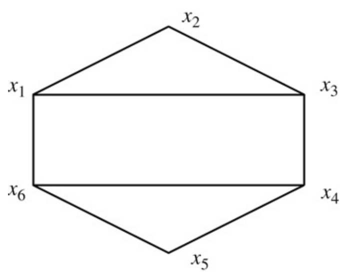
A λ-fold *m-cycle system* of *order n* is a pair (X, C), where X is a finite set of n elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_n, the complete multigraph with vertex set X, where every pair of vertices is joined by λ edges. In this case, $|C| = \lambda n(n-1)/2m$. When $\lambda = 1$, we will simply say *m-cycle system*. A 3-cycle is also called a *triple* and so a λ-fold 3-*cycle system* will also be called a λ-fold *triple system*. When $\lambda = 1$, we have the well-known definition of a *Steiner triple system* (or, simply, a *triple system*).

Fairly recently the spectrum (i.e. the set of all n such that a *m-cycle* system of order n exists) has been determined to be [1,3]:

$$\begin{cases} (1)\ n \geq m, \quad \text{if } n > 1, \\ (2)\ n \text{ is odd, and} \\ (3)\ \dfrac{n(n-1)}{2m} \text{ is an integer.} \end{cases}$$

The spectrum for λ-fold *m-cycle system* for $\lambda \geq 2$ is still an open problem.

The graph given below



is called a *hexagon quadrangle* and will be denoted by $[(x_1, x_2, x_3), (x_4, x_5, x_6)]$.

A *hexagon quadrangle system* of *order n* and *index ρ* [$\mathrm{HQS}_\rho(n)$, or simply $\mathrm{HQS}(n)$ when $\rho = 1$] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint hexagon quadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X.

☆ Lavoro eseguito nell'ambito del GNSAGA [INDAM] e del MIUR [PRIN (2005) "*Strutture geometriche, combinatoria, loro Applicazioni*"].

E-mail address: lucia.gionfriddo@dmi.unict.it.

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A *kite* is a graph obtained from the complete graph on 4 vertices K_4, deleting a path of length 2: so, if the vertices are x, y, z, t, the edges are $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x, t\}$. We will denote such a *kite* by $[(x, y, z), t]$. Observe that, if we consider a *hexagon quadrangle* $q = [(x_1, x_2, x_3), (x_4, x_5, x_6)]$, it is possible to partition it into the two kites $q(k') = [(x_1, x_2, x_3), x_6]$, $q(k'') = [(x_4, x_5, x_6), x_3]$.

A *kite system* of order n and index μ is a pair (X, K), where X is a finite set of n elements, called vertices, and K is a decomposition of the complete graph μK_n in graphs all isomorphic to a kite.

A *hexagon quadrangle system* (X, H) of *order n* and *index* ρ is said to be *kite nesting* if for every hexagon quadrangle $q \in H$ there exists at least a kite $q(k) \in \{q(k'), q(k'')\}$ such that the collection Q of all these *kites* $q(k)$ form a *kite system* of index μ. This kite system is said to be nested in (X, H). We will call it a *hexagon kite system*of *order n* and *indices* (ρ, μ) [briefly, *kite nesting* HQS$(n; \rho, \mu)$]. If K is the family of all the kites $q(k'), q(k'')$ contained in the hexagon quadrangles $q \in H$, we observe that also the family $Q^c = K - Q$ forms a *kite system* of index $\rho - \mu$. If, for every hexagon quadrangle $q \in H$, both families of kites $Q_{k'} = \{q(k') : q \in H\}$, $Q_{k''} = \{q(k'') : q \in H\}$ form a kite system of index μ, we will say that the HQS is a *hexagon bi-kite system* [briefly, HKS$(n; \rho, \mu)$] and, also, that it is a bi-kite nesting system.

In this paper we completely determine the spectrum of HKS(n), for $\mu = 4h$ and $\rho = 8h$, h a positive integer. Exactly, we prove that: "*For each* $h \in N$, *the spectrum of* HKS$(n; 8h, 4h)$ *is the set of all positive integers* n, $n \geq 7$".

In what follows we will use the following function, for every positive integers i, p:

$$\left[\frac{i}{2}\right]_p = \begin{cases} \dfrac{i}{2} & \text{if } i \text{ is even} \\ \dfrac{p+i}{2} & \text{if } i \text{ is odd.} \end{cases}$$

2. Necessary conditions for HKS

In this section we prove some necessary conditions for the existence of HKSs.

Lemma 2.1. *Let* (X, H) *be an* HKS$(n; \rho, \mu)$. *Then:* $\rho = 2\mu$.

Proof. Let (X, H) be a HKS$(n; \rho, \mu)$and let (X, K) be the *kite system* of index μ, nested in it.

It follows that

$$|H| = |K|,$$
$$|H| = \binom{n}{2} \cdot \frac{\rho}{8}, \qquad |K| = \binom{n}{2} \cdot \frac{\mu}{4}.$$

It follows: $\rho = 2\mu$. □

Lemma 2.2. *Let* (X, H) *be a* HKS(n) *with associated indices* (ρ, λ, μ). *Then:*

(i) $(\rho, \mu) = (2, 1)$ *implies:* $n \equiv 0, 1 (\mathrm{mod}\, 8)$;
(ii) $(\rho, \mu) = (4, 2)$ *implies:* $n \equiv 0, 1 (\mathrm{mod}\, 4)$, $n \geq 6$;
(iii) $(\rho, \mu) = (6, 3)$ *implies:* $n \equiv 0, 1 (\mathrm{mod}\, 4)$, $n \geq 6$;
(iv) $(\rho, \mu) = (8, 4)$ *implies:* $n \equiv 0, 1 (\mathrm{mod}\, 2)$, $n \geq 6$.

Proof. It is sufficient to consider that

$$|K| = \binom{n}{2} \cdot \frac{\mu}{4}. \quad \square$$

3. Existence and nonexistence results

In this section we will examine the existence of HKS$(n; \rho, \mu)$ for $(\rho, \mu) = (8, 4)$ and n a prime number or an odd number not divisible by three or five.

Proposition 3.1. *For every prime number* p, $p \geq 7$, *there exists a* HKS(p) *with indices* $(\rho, \mu) = (8, 4)$.