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Singletons and adjacencies of set partitions of type B

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ABSTRACT

We show that the joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over set partitions of type B_n without a zero block, in analogy with the result of Callan for ordinary partitions.

1. Introduction

The main objective of this paper is to give a type-B analogue of a property of set partitions discovered by Bernhart [1], namely, that the number s_n of partitions of $[n] = \{1, 2, \dots, n\}$ without singletons is equal to the number a_n of partitions of $[n]$ for which no block contains two adjacent elements i and $i + 1$ modulo n. In fact, it is easy to show that s_n and a_n have the same formula by the principle of inclusion–exclusion. Bernhart gave a recursive proof of the fact that $s_n = a_n$ by showing that $s_n + s_{n+1} = B_n$ and $a_n + a_{n+1} = B_n$, where B_n denotes the Bell number, namely, the number of partitions of $[n]$. As noted by Bernhart, there may be no simple way to bring the set of partitions of $[n]$ without singletons and the set of partitions of $[n]$ without adjacencies into a one-to-one correspondence.

From a different perspective, Callan [4] found a bijection in terms of an algorithm that interchanges singletons and adjacencies. Indeed, Callan has established a stronger statement that the joint distribution of the number of singletons and the number of adjacencies is symmetric over the set of partitions of $[n]$. While Callan's proof is purely combinatorial, we feel that there is still some truth in Bernhart's remark.

The study of singletons and adjacencies of partitions goes back to Kreweras [9] for noncrossing partitions. Kreweras has shown that the number of noncrossing partitions of $[n]$ without singletons equals the number of noncrossing partitions of $[n]$ without adjacencies. Bernhart [1] found a combinatorial proof of this assertion. Deutsch and Shapiro [6] considered noncrossing partitions of $[n]$ without visible singletons, and showed that such partitions are enumerated by the Fine number. Here, a visible singleton of a partition means a singleton not covered by any arc in the linear representation. Canfield [5] has shown that the average number of singletons in a partition of $[n]$ is an increasing function of n. Biane [2] has derived a bivariate generating function for the number of partitions of $[n]$ containing a given number of blocks but no singletons. Knuth [8] proposed the problem of finding the generating function for the number of partitions of $[n]$ with a given number of

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blocks but no adjacencies. The generating function has been found by several problem solvers; see Lossers [10], for example. The sequence of the numbers s_n is listed as the entry A000296 in Sloane [12].

The lattice of ordinary set partitions can be regarded as the intersection lattice for the hyperplane arrangement corresponding to the root system of type A; see Björner and Brenti [3] or Humphreys [7]. Type-B set partitions are a generalization of ordinary partitions from this point of view; see Reiner [11]. More precisely, ordinary set partitions encode the intersections of hyperplanes in the hyperplane arrangement for the type-A root system, while the intersections of subsets of hyperplanes from the type-B hyperplane arrangement can be encoded by type-B set partitions. A partition of type B_n is a partition π of the set

$$[\pm n] = \{1, 2, \ldots, n, -1, -2, \ldots, -n\}$$

such that, for any block B of π, $-B$ is also a block of π, and such that π has at most one block, called the zero block, which is of the form $\{i_1, i_2, \ldots, i_k, -i_1, -i_2, \ldots, -i_k\}$.

It is natural to ask whether there exists a type-B analogue of Bernhart's theorem and a type-B analogue of Callan's algorithm. We give the peeling and patching algorithm which implies the symmetric distribution of the number of singleton pairs and the number of adjacency pairs for type-B partitions without a zero block. Moreover, we can transform the bijection into an involution.

2. The peeling and patching algorithm

In this section, we give a type-B analogue of Callan's symmetric distribution of singletons and adjacencies over set partitions. We find that Callan's algorithm can be extended into the type-B case. This type-B algorithm will be called the peeling and patching algorithm.

Let π be a B_n-partition. We call $\pm i$ a singleton pair of π if π contains a block $\{i\}$, and call $\pm(j, j+1)$ an adjacency pair of π if j and $j+1$ (modulo n) lie in the same block of π. Denote the number of singleton pairs (resp., adjacency pairs) of π by s_π (resp., a_π). For example, let $n = 12$ and

$$\pi = \big\{\pm\{1\}, \pm\{2\}, \pm\{3, 11, 12\}, \pm\{4, -7, 9, 10\}, \pm\{5, 6, -8\}\big\}. \tag{2.1}$$

Then we have $s_\pi = 2$ and $a_\pi = 3$.

Denote by V_n the set of B_n-partitions without a zero block. The following theorem is the main result of this paper.

Theorem 2.1. *The joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over B_n-partitions without a zero block. In other words, let*

$$P_n(x, y) = \sum_{\pi \in V_n} x^{s_\pi} y^{a_\pi};$$

then we have $P_n(x, y) = P_n(y, x)$.

For example, there are three B_2-partitions without a zero block:

$$\big\{\pm\{1\}, \pm\{2\}\big\}, \qquad \big\{\pm\{1, 2\}\big\}, \qquad \big\{\pm\{1, -2\}\big\}.$$

So $P_2(x, y) = x^2 + y^2 + 1$. Moreover,

$$P_3(x, y) = (x^3 + y^3) + 3xy + 3(x + y),$$
$$P_4(x, y) = (x^4 + y^4) + 4(x^2y + xy^2) + 8(x^2 + y^2) + 8xy + 4(x + y) + 7.$$

Recall that Bernhart showed that the number of partitions of $[n]$ without singletons equals the number of partitions of $[n]$ without adjacencies. As a type-B analogue of this result, we have the following consequence of Theorem 2.1.

Corollary 2.2. *The number of B_n-partitions containing no zero block and no singleton pairs equals the number of B_n-partitions containing no zero block and no adjacency pairs.*

To prove Theorem 2.1, we shall construct a map $\psi: V_n \to V_n$, called the peeling and patching algorithm, such that, for any B_n-partition π without a zero block, $s_\pi = a_{\psi(\pi)}$ and $a_\pi = s_{\psi(\pi)}$. To describe this algorithm, we need a more general setting. Let

$$S = \{\pm t_1, \pm t_2, \ldots, \pm t_r\}$$

be a subset of $[\pm n]$, where $0 < t_1 < t_2 < \cdots < t_r$. Let π be a partition of S. We call π a *symmetric partition* if, for any block B of π, $-B$ is also a block of π. Similarly, we call $\pm t_i$ a *singleton pair* of π if π contains a block $\{t_i\}$, and call $\pm(t_j, t_{j+1})$ an *adjacency pair* of π if t_j and t_{j+1} are contained in the same block. Moreover, we identify t_{r+1} with t_1. We call $\pm t_j$ (resp., $\pm t_{j+1}$) a *left-point pair* (resp., *right-point pair*) if $\pm(t_j, t_{j+1})$ is an adjacency pair. For example, for the case $r = 1$, the partition $\pi = \big\{\pm\{t_1\}\big\}$ contains exactly one singleton pair $\pm t_1$ and one adjacency pair $\pm(t_1, t_1)$.