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The base of a primitive, nonpowerful sign pattern with exactly d nonzero diagonal entries

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ABSTRACT

In [J.Y. Shao, L.H. You, H.Y. Shan, Bound on the base of irreducible generalized sign pattern matrices, Linear Algebra Appl. 427 (2007) 2–3, 285–300], Shao, You and Shan extended the concept of the base from powerful sign pattern matrices to nonpowerful (generalized) sign pattern matrices. It is well known that the properties of the power sequences of different classes of sign pattern matrices may be very different. In this paper, we consider the base set of the primitive nonpowerful square sign pattern matrices of order n with exactly d (with $d \geq 1$) nonzero diagonal entries. The base set is shown to be $\{2, 3, \dots, 3n - d - 1\}$. The extremal sign pattern matrices with both the least number $n + d$ nonzero entries and the maximum base $3n - d - 1$ are characterized.

1. Introduction

We adopt the standard conventions, notations and definitions for sign patterns and generalized sign patterns, their entries, arithmetics and powers, and in particular, for walks in the corresponding signed digraphs. The reader who is not familiar with these matters should consult the 2008 paper by Cheng [1] or the 2010 paper by Li et al. [7].

The sign pattern of a real matrix A, denoted by $\mathrm{sgn}(A)$, is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. Notice that in the computation of the entries of the power A^k, an "*ambiguous sign*" may arise when we add a positive sign to a negative sign. So a new symbol "#" has been introduced to denote the ambiguous sign in [6].

For convenience, we call the set $\Gamma = \{0, 1, -1, \#\}$ the generalized sign set, and we adopt the standard rules for sums and products involving # (see [1] or [14], for example).

Definition 1.1. Let A be a square sign pattern matrix of order n with power sequence $A, A^2, \dots$. Because there are only 4^{n^2} different generalized sign pattern matrices of order n, there must be repetitions in the power sequence of A. Suppose that $A^l = A^{l+p}$ is the first pair of powers that are repeated in the sequence. Then l is called the *generalized base* (or simply base) of A, and is denoted by $l(A)$. The least positive integer p such that $A^l = A^{l+p}$ holds for $l = l(A)$ is called the *generalized period* (or simply period) of A, and is denoted by $p(A)$. For a square $(0, 1)$-Boolean matrix A, $l(A)$ is also known as the convergence index of A, denoted by $k(A)$.

In 1994, Li et al. [6] extended the concept of the base (or convergence index) and period from nonnegative matrices to sign pattern matrices. They defined powerful and nonpowerful for sign pattern matrices, gave a sufficient and necessary condition that an irreducible sign pattern matrix is powerful, and also gave a condition for the nonpowerful case.

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Definition 1.2. A square sign pattern matrix A is powerful if all the powers $A^1, A^2, A^3, \ldots$ are unambiguously defined, namely, there is no # in A^k $(k = 1, 2, \ldots)$. Otherwise, A is called nonpowerful.

In this paper, for a sign pattern matrix A, we denote by $|A|$ the nonnegative matrix obtained from A by replacing a_{ij} with $|a_{ij}|$.

Definition 1.3. An irreducible (0, 1)-Boolean matrix A is primitive if there exists a positive integer k such that all the entries of A^k are nonzero; the least such k is called the primitive index of A, denoted by $\exp(A) = k$. A square sign pattern matrix A is called primitive if $|A|$ is primitive, and the primitive index of A, denoted by $\exp(A)$, equals $\exp(|A|)$.

It is well known that graph theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph theoretical concepts.

Definition 1.4. Let A be a square sign pattern matrix of order n. The *associated digraph* of A, denoted by $D(A)$, has vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j)|a_{ij} \neq 0\}$. The *associated signed digraph* of A, denoted by $S(A)$, is obtained from $D(A)$ by assigning the sign of a_{ij} to arc (i, j) for all i and j. Let S be a signed digraph of order n and let A be a square sign pattern matrix of order n; A is called the *associated sign pattern matrix* of S if $S(A) = S$. The associated sign pattern matrix of a signed digraph S is always denoted by $A(S)$. Note that $D(A) = D(|A|)$, so $D(A)$ is also called the *underlying digraph* of the associated signed digraph of A, or is simply called the *underlying digraph* of A. We always denote by $D(A(S))$ or $|S|$ the underlying digraph of a signed digraph S. Sometimes, $|A(S)|$ is called the associated or underlying matrix of signed digraph S.

In this paper, we permit loops but no multiple arcs in a signed digraph. Let $W = v_0e_1v_1e_2 \cdots e_kv_k$ $(e_i = (v_{i-1}, v_i),\ 1 \le i \le k)$ be a directed walk of signed digraph S. The sign of W, denoted by $\mathrm{sgn}(W)$, is $\prod_{i=1}^{k} \mathrm{sgn}(e_i)$. Sometimes a directed walk can be denoted simply by $W = v_0v_1 \cdots v_k$, $W = (v_0, v_1, \ldots, v_k)$ or $W = e_1e_2 \cdots e_k$ if there is no ambiguity. When there is no ambiguity, a directed walk, a directed path or a directed cycle will be called a walk, a path or a cycle. A walk is called a *positive walk* if its sign is positive, and a walk is called a *negative walk* if its sign is negative. If p is a positive integer and if C is a cycle, then pC denotes the walk obtained by traversing Cp times. If a cycle C passes through the end vertex of W, $W \bigcup pC$ denotes the walk obtained by going along W and then going around the cycle Cp times; $pC \bigcup W$ is similarly defined.

Definition 1.5. Assume that W_1 and W_2 are two directed walks in signed digraph S. They are called a pair of SSSD walks if they have the same initial vertex, the same terminal vertex and the same length, but they have different sign.

From [6] or [14], we know that a signed digraph S is powerful if only if there is no pair of SSSD walks in S. Otherwise, S is nonpowerful.

Definition 1.6. A strongly connected digraph S is primitive if there exists a positive integer k such that, for all vertices $v_i, v_j \in V(S)$ (not necessarily distinct), there exists a directed walk of length k from v_i to v_j. The least such k is called the primitive index of S, and is denoted by $\exp(S)$. Let S be a primitive digraph. The least l such that there is a directed walk of length t from v_i to v_j for any integer $t \ge l$ is called the local primitive index from v_i to v_j, denoted by $\exp_S(v_i, v_j) = l$. Similarly, $\exp_S(v_i) = \max_{v_j \in V(S)}\{\exp_S(v_i, v_j)\}$ is called the local primitive index at v_i, so $\exp(S) = \max_{v_i \in V(S)}\{\exp_S(v_i)\}$.

For a square sign pattern A, let $W_k(i, j)$ denote the set of walks of length k from vertex i to vertex j in $S(A)$. Notice that the entry $(A^k)_{ij}$ of A^k satisfies $(A^k)_{ij} = \sum_{W \in W_k(i,j)} \mathrm{sgn}(W)$; then we have

(1) $(A^k)_{ij} = 0$ if and only if there is no walk of length k from i to j in $S(A)$ (i.e., $W_k(i, j) = \phi$);
(2) $(A^k)_{ij} = 1$ (or -1) if and only if $W_k(i, j) \neq \phi$ and all walks in $W_k(i, j)$ have the same sign 1 (or -1);
(3) $(A^k)_{ij} = \#$ if and only if there is a pair of SSSD walks of length k from i to j.

So the associated signed digraph can be used to study the properties of the power sequence of a sign pattern matrix, and the signed digraph is taken as the tool in this paper. In matrix theory, a primitive matrix must be a nonnegative real matrix. From the relation between sign pattern matrices and signed digraphs, for a primitive signed digraph S, we have $\exp(S) = \exp(|A(S)|)$. Hence it is logical to define a sign pattern A to be primitive if $|A|$ is primitive, and to define $\exp(A) = \exp(D(A)) = \exp(|A|)$ if A is primitive.

Definition 1.7. A signed digraph S is primitive nonpowerful if there exists a positive integer l such that, for any integer $t \ge l$, there is a pair of SSSD walks of length t from any vertex v_i to any vertex $v_j (v_i, v_j \in V(S))$. The least such l is called the base of S, denoted by $l(S)$. Let S be a primitive nonpowerful signed digraph of order n. For $u, v \in V(S)$, the local base from u to v, denoted by $l_S(u, v)$, is defined to be the least integer k such that there is a pair of SSSD walks of length t from u to v for any integer $t \ge k$. The local base at a vertex $u \in V(S)$ is defined to be $l_S(u) = \max_{v \in V(S)}\{l_S(u, v)\}$. So $l(S) = \max_{u \in V(S)} l_S(u) = \max_{u,v \in V(S)} l_S(u, v)$.