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The signless Laplacian spectral radius of bicyclic graphs with prescribed degree sequences ☆

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ABSTRACT

In this paper, we characterize all extremal connected bicyclic graphs with the largest signless Laplacian spectral radius in the set of all connected bicyclic graphs with prescribed degree sequences. Moreover, the signless Laplacian majorization theorem is proved to be true for connected bicyclic graphs. As corollaries, all extremal connected bicyclic graphs having the largest signless Laplacian spectral radius are obtained in the set of all connected bicyclic graphs of order n (resp. all connected bicyclic graphs with a specified number of pendant vertices, and all connected bicyclic graphs with given maximum degree).

1. Introduction

In this paper, only connected simple graphs are considered. Let $G = (V, E)$ be a connected simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. If $|E| = |V| + c - 1$, then $G = (V, E)$ is called a c-cyclic graph, where c is a nonnegative integer. Especially, if $c = 0$ (resp. $c = 1$ and 2), then G is called a tree (resp. unicyclic and bicyclic graphs). Let $u, v \in V(G)$. If u is adjacent to v in G, then v is called the neighbor of u. Let $N(u)$ denote the neighbor set of u. Then $d(u) = d_G(u) = |N(u)|$ is called the degree of u in G. A non-increasing sequence of nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is called graphic if there exists a graph G having π as its vertex degree sequence (π is also called the degree sequence of G). For all other definitions, notations and terminologies on the spectral graph theory, not given here, see e.g. [1,4,6].

Denote by $D(G) = \mathrm{diag}(d(u), u \in V)$ the diagonal matrix of vertex degrees of G and $A(G)$ the $(0, 1)$-adjacency matrix of G. If G is connected, then the matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G (also called the unoriented Laplacian matrix of G in [12,18]), which may be first introduced in book [4] without giving a name. The signless Laplacian matrices of graphs have received much attention in recent years (see e.g. [3,5,7–10,12,13,18]). The largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G and is denoted by $\mu(G)$. Note that $Q(G)$ is an irreducible nonnegative matrix, then there exists only one unit positive eigenvector $f = (f(u), v \in V(G))$ with $\|f\| = 1$ such that $Q(G)f = \mu(G)f$. Such an eigenvector f is called the Perron vector of $Q(G)$, and $f(u)$ is called the μ-weight of vertex u [15].

For a prescribed graphic degree sequence π, let $\mathscr{G}_\pi = \{G \mid G \text{ is simple, connected and with } \pi \text{ as its degree sequence}\}$. Motivated by the Brualdi–Solheid problem [2], Zhang put forward the determination of graphs maximizing (or minimizing) the signless Laplacian spectral radius in the set $\mathscr{G}_\pi$ [21]. It has been proved in [20,21] that for a given degree sequence π of trees (resp. unicyclic graphs), there exists a unique tree (resp. unicyclic graph) that has the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$.

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Let $\mathscr{B}_\pi$ denote the class of all bicyclic graphs with bicyclic degree sequence π. In this paper, we determine the unique maximal graph in $\mathscr{B}_\pi$ with respect to the signless Laplacian spectral radius of $G \in \mathscr{B}_\pi$.

Moreover, recall the notion of majorization. Let $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ be two non-increasing sequences. If $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d'_i$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$, then the sequence π is said to be majorized by the sequence π', which is denoted by $\pi \lhd \pi'$.

Let π and π' be two different tree (resp. unicyclic) degree sequences with the same order. Let G and G' be the trees (resp. unicyclic graphs) with the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$ and $\mathscr{G}_{\pi'}$, respectively. In [20] (resp. [21]), the author proved that if $\pi \lhd \pi'$, then $\mu(G) < \mu(G')$. Such a theorem is called the signless Laplacian majorization theorem for trees (resp. unicyclic graphs). In Section 3, the signless Laplacian majorization theorem is proved to be true for bicyclic graphs. As corollaries, all extremal bicyclic graphs having the largest signless Laplacian spectral radius are obtained in the set of all bicyclic graphs of order n (resp. all bicyclic graphs with a specified number of pendant vertices, all bicyclic graphs with given maximum degree).

2. Preliminaries

In this section, we introduce some definitions and lemmas which are useful in the presentations and proofs of our main results.

Let G be a graph with a root vertex $v_1 \in V(G)$. Denote by $\mathrm{dist}(v, v_1)$ the distance between $v \in V(G)$ and v_1. Besides, the distance $\mathrm{dist}(v, v_1)$ is called the height of vertex v in G, denoted by $h(v) = \mathrm{dist}(v, v_1)$.

Definition 2.1 (*[21]*). Let $G = (V, E)$ be a graph with a root $v_1 \in V(G)$. A well-ordering $\prec$ of the vertices is called a breadth-first-search ordering (BFS-ordering for short) if the following hold for all vertices $u, v \in V$:

(1) $u \prec v$ implies $h(u) \leq h(v)$;
(2) $u \prec v$ implies $d(u) \geq d(v)$;
(3) suppose $uv, xy \in E$, $uy, xv \notin E$ with $h(u) = h(x) = h(v) - 1 = h(y) - 1$. If $u \prec x$, then $v \prec y$.

We call a graph that has a BFS-ordering of its vertices a BFS-graph.

Lemma 2.2 (*[21]*). *Let $G = (V, E)$ be a simple connected graph having the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$ and f be the Perron vector of $Q(G)$. If $V = \{v_1, \ldots, v_n\}$ satisfies $f(v_i) \geq f(v_j)$ for $i < j$ (i.e., the vertices of $V(G)$ are denoted with respect to $f(v)$ in non-increasing order), then $d(v_i) \geq d(v_j)$ for $i < j$. Moreover, if $f(v_i) = f(v_j)$, then $d(v_i) = d(v_j)$.*

Corollary 2.3. *Let $G = (V, E)$ be a simple connected graph having the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$ and f be the Perron vector of $Q(G)$. If $d(u) > d(v)$, then $f(u) > f(v)$, where $u, v \in V$.*

Proof. Suppose $f(v) \geq f(u)$. By Lemma 2.2, $d(v) \geq d(u)$, a contradiction. □

Lemma 2.4 (*[21]*). *Let $G = (V, E)$ be a simple connected graph having the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$. Then G has a BFS-ordering consistent with the Perron vector f of $Q(G)$ in such a way that $u \prec v$ implies $f(u) \geq f(v)$.*

From the Perron–Frobenius Theorem of nonnegative matrices [15], we have

Lemma 2.5 (*[15]*). *If H is a proper connected subgraph of a connected simple graph G, then $\mu(H) < \mu(G)$.*

Suppose that $uv \in E(G)$ (resp. $uv \notin E(G)$). Denote by $G - uv$ (resp. $G + uv$) the graph obtained from G by deleting (resp. adding) the edge uv.

Lemma 2.6 (*[20]*). *Let $G = (V, E) \in \mathscr{G}_\pi$ and f be a Perron vector of $Q(G)$.*

(1) *Suppose $uw_i \in E$ and $vw_i \notin E$ for $i = 1, \ldots, k$. Let $G' = G + vw_1 + \cdots + vw_k - uw_1 - \cdots - uw_k$. If G' is connected and $f(v) \geq f(u)$, then $\mu(G') > \mu(G)$.*
(2) *If $d(u) > d(v)$ and $f(u) \leq f(v)$, then there exists a connected graph $G' \in \mathscr{G}_\pi$ such that $\mu(G') > \mu(G)$.*

Lemma 2.7 (*[20]*). *Let $G = (V, E) \in \mathscr{G}_\pi$ and f be a Perron vector of $Q(G)$. Assume that $v_1u_1, v_2u_2 \in E$, and $v_1v_2, u_1u_2 \notin E$. Let $G' = G + v_1v_2 + u_1u_2 - v_1u_1 - v_2u_2$. If G' is connected and $(f(v_1) - f(u_2))(f(v_2) - f(u_1)) \geq 0$, then $\mu(G') \geq \mu(G)$. Moreover, the last equality holds if and only if $f(v_1) = f(u_2)$ and $f(v_2) = f(u_1)$.*

Corollary 2.8. *Let $G = (V, E)$ be a simple connected graph that has the largest signless Laplacian spectral radius in $\mathscr{G}_\pi$ and f be a Perron vector of $Q(G)$. Assume that $v_1u_1, v_2u_2 \in E$, and $v_1v_2, u_1u_2 \notin E$. Let $G' = G + v_1v_2 + u_1u_2 - v_1u_1 - v_2u_2$. Then the degree sequence of G' is π. Moreover,*

(1) *if G' is connected and $f(v_1) > f(u_2)$, then $f(v_2) < f(u_1)$;*
(2) *if G' is connected and $f(v_1) = f(u_2)$, then $f(v_2) = f(u_1)$.*