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Parity vertex colouring of plane graphs

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ABSTRACT

A proper vertex colouring of a 2-connected plane graph G is a *parity vertex colouring* if for each face f and each colour c, either no vertex or an odd number of vertices incident with f is coloured with c. The minimum number of colours used in such a colouring of G is denoted by $\chi_p(G)$.

In this paper, we prove that $\chi_p(G) \leq 118$ for every 2-connected plane graph G.

1. Introduction

The four colour problem [1,15] has served as a starting point and motivation for many interesting problems. The solution of this problem is known as the four colour theorem.

It follows from the four colour theorem that the vertices of any plane triangulation T can be coloured with at most four colours in such a way that vertices of the same face receive different colours. This simple observation led Ore and Plummer [13] to introduce a *cyclic colouring*. A cyclic colouring of a plane graph G is a vertex colouring in which any two vertices incident with the same face receive different colours. The minimum number of colours in any cyclic colouring of a plane graph G is called the *cyclic chromatic number* of G and is denoted by $\chi_c(G)$. If a graph G is 2-connected, then any face f of G is incident with $\deg(f)$ vertices. Hence, $\chi_c(G)$ is naturally lower bounded by $\Delta^*(G)$, the maximum face size of G. Sanders and Zhao [16] proved that $\chi_c(G) \leq \left\lceil \frac{5\Delta^*(G)}{3} \right\rceil$ for any 2-connected plane graph G. On the other hand, for any $d \geq 4$ there is a 2-connected plane graph G_d satisfying $\Delta^*(G_d) = d$ and $\chi_c(G_d) = \left\lfloor \frac{3\Delta^*(G)}{2} \right\rfloor$. It is conjectured [12] that $\chi_c(G) \leq \left\lfloor \frac{3\Delta^*(G)}{2} \right\rfloor$ for any 2-connected plane graph G.

Plummer and Toft [14] proposed the conjecture that if G is a 3-connected plane graph, then $\chi_c(G) \leq \Delta^*(G) + 2$. This conjecture is true for plane triangulations (by the four colour theorem) and for 3-connected plane graphs with $\Delta^*(G) \geq 18$, see Horňák and Jendroľ [10] for $\Delta^*(G) \geq 24$ and Horňák and Zlámalová [11] for the remaining cases. For $\Delta^*(G) = 4$, this conjecture is verified by Borodin [3]. Enomoto et al. [9] obtained for $\Delta^*(G) \geq 60$ even a stronger result, namely $\chi_c(G) \leq \Delta^*(G) + 1$. The best known general result is the inequality $\chi_c(G) \leq \Delta^*(G) + 5$ of Enomoto and Horňák [8].

Another motivation for this paper comes from a parity colouring concept introduced in recent papers of Bunde, Milans, West, and Wu. In [4,5] they introduced a *strong parity edge colouring* of graphs being an edge colouring of a graph G such that each open walk in G uses at least one colour an odd number of times. The minimum number of colours in a strong

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parity edge colouring of G is the strong parity edge chromatic number $p(G)$. The exact value of $p(K_n)$ for complete graphs is determined in [4]. As mentioned in [5], the problem of determining $p(G)$ is NP-hard even when G is a tree.

The authors of [7] focused on facial walks of plane graphs. They introduced a *facial parity edge colouring*, which is an edge colouring such that no two consecutive edges of a facial walk of any face f receive the same colour and for each face f and each colour c, either no edge or an odd number of edges incident with f is coloured with c. The problem is to determine the minimum number of colours used in such a colouring. This number is called the *facial parity chromatic index*. Note that the facial parity chromatic index depends on the embedding of the graph. The authors of [7] proved that each 2-edge-connected plane multigraph has a facial parity chromatic index at most 20. Moreover, the upper bound is at most 12 for any 3-edge-connected plane multigraph, and for a 4-edge-connected plane multigraph the upper bound is at most 9.

In this paper, we investigate a *parity vertex colouring* of 2-connected plane graphs which can be considered as a relaxation of the cyclic colouring. A proper vertex colouring of a 2-connected plane graph is a parity vertex colouring if for each face f and each colour c, either no vertex or an odd number of vertices incident with f is coloured with c. The minimum number of colours in any parity vertex colouring of a 2-connected plane graph G is called the *parity chromatic number* of G and is denoted by $\chi_p(G)$.

If $\chi_0(G)$ denotes the (usual) chromatic number of a 2-connected plane graph G, then immediately from the definitions we have

$$\chi_0(G) \leq \chi_p(G) \leq \chi_c(G).$$

Notice that for plane triangulations, proper (usual) colourings, cyclic colourings, and parity vertex colourings coincide. Moreover, for 2-connected plane graphs with maximum face size at most 5, cyclic colourings and parity vertex colourings coincide too.

The parameter $\chi_p(G)$ has been introduced in Czap and Jendroľ [6], where the authors have conjectured the existence of a constant K such that $\chi_p(G) \leq K$ for every 2-connected plane graph G.

The main result of this paper is the proof of this conjecture, namely we show that $K \leq 118$.

2. Notation

Throughout this paper, we use the standard terminology according to Bondy and Murty [2]. However, we recall some frequently used terms.

A *planar graph* is a graph which can be embedded in the plane. A *plane graph* is a fixed embedding of a planar graph.

In this paper, we consider 2-connected plane graphs; they may contain parallel edges but loops are not allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E, and the face set F. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident with v. A k-vertex is a vertex of degree k. The *size* of a face f is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of f. The size of f is denoted by $\deg(f)$. A k-face is a face of size k. Two faces are *adjacent* if they share an edge.

Given a graph G and one of its edges, say $e = uv$, the *contraction* of e is the operation on G which consists in replacing u and v by a new vertex adjacent to all the former neighbours of u and v, and removing the loop corresponding to the edge e. (We keep multiple edges if they occur). The resulting graph is denoted by $G\%e$. Analogously, we define the contraction of the set of edges $S = \{e_1, \ldots, e_k\}$ and we denote it by $G\%\{e_1, \ldots e_k\}$ or $G\%S$.

Let H be a subgraph of G. Then, the graph $G \setminus H$ is defined as a graph obtained from G by deleting the edges in $E(H)$. (We delete isolated vertices if they occur).

A *vertex k-colouring* of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. A vertex colouring is called *proper* if no two adjacent vertices have the same colour.

A proper vertex colouring φ is a *parity vertex* (*PV*) *colouring* of a 2-connected plane graph G if for each face f and each colour c, either no vertex or an odd number of vertices incident with f is coloured with c.

3. Main result

The main result of this paper is the following theorem.

Theorem 1. *Let G be a 2-connected plane graph. Then*

$$\chi_p(G) \leq 118.$$

The remainder of this paper contains the proof of this theorem. The proof uses a discharging method.

Suppose there is a counterexample to Theorem 1. Let G be a counterexample with the minimum number of vertices, say n, and the minimum number of edges among all counterexamples on n vertices.

First, we prove several structural properties of G.

We say that a face f is *small* if $2 \leq \deg(f) \leq 59$; otherwise, it is called *big*.

3.1. Reducible configurations

In this section, we deal with subgraphs H of G such that a parity vertex colouring of $G \setminus H$ or $G\%H$ using at most 118 colours can be extended to a parity vertex colouring of G using at most 118 colours. Thus, these subgraphs cannot occur in