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Posets associated with subspaces in a d-bounded distance-regular graph

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ABSTRACT

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 3$. A regular strongly closed subgraph of Γ is said to be a subspace of Γ. For $x \in X$, let $P(x)$ be the set of all subspaces of Γ containing x. For each $i = 1, 2, \ldots, d-1$, let Δ_0 be a fixed subspace with diameter $d - i$ in $P(x)$, and let

$$\mathscr{L}(d, i) = \{\Delta \in P(x) \mid \Delta + \Delta_0 = \Gamma, d(\Delta) = d(\Delta \cap \Delta_0) + i\} \cup \{\emptyset\}.$$

If we define the partial order on $\mathscr{L}(d, i)$ by ordinary inclusion (resp. reverse inclusion), then $\mathscr{L}(d, i)$ is a finite poset, denoted by $\mathscr{L}_O(d, i)$ (resp. $\mathscr{L}_R(d, i)$). In the present paper we show that both $\mathscr{L}_O(d, i)$ and $\mathscr{L}_R(d, i)$ are atomic, and compute their characteristic polynomials.

1. Introduction

In this section we recall some terminologies and definitions about posets [1,2] and d-bounded distance-regular graphs.

Let P be a poset. For $a, b \in P$, we say *a covers b*, denoted by $b \lessdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. Let P be a finite poset with the minimum element, denoted by 0. By a *rank function* on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b \lessdot a$. Observe the rank function is unique if it exists. P is said to be *ranked* whenever P has a rank function. Let P be a finite poset with 0. By an *atom* we mean an element of P covering 0. We say P is *atomic* if each element in $P \setminus \{0\}$ is the *join* of atoms in P.

Let P be a ranked poset with 0 and the maximum element, denoted by 1. The polynomial

$$\chi(P, y) = \sum_{a \in P} \mu(0, a) y^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P, where μ is the Möbius function on P and r is the rank function of P.

Let P and P' be two posets. If there exists a bijection σ from P to P' such that, for all $a, b \in P$, $a < b$ if and only if $\sigma(a) < \sigma(b)$, then σ is said to be an isomorphism from P to P'. In this case, we call P is isomorphic to P'. It is well known that two isomorphic posets have the same Möbius function.

Let $\Gamma = (X, R)$ be a connected regular graph. For vertices u and v of Γ, $\partial(u, v)$ denotes the *distance* between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $d(\Gamma)$. For vertices u and v at distance i, define

$$C(u, v) = C_i(u, v) = \{w \mid \partial(u, w) = i - 1, \partial(w, v) = 1\},$$
$$A(u, v) = A_i(u, v) = \{w \mid \partial(u, w) = i, \partial(w, v) = 1\}.$$

For the cardinalities of these sets we use lower case letters $c_i(u, v)$ and $a_i(u, v)$.

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A connected regular graph Γ with diameter d is said to be *distance-regular* if $c_i(u, v)$ and $a_i(u, v)$ depend only on i for all $1 \le i \le d$. The reader is referred to [3] for general theory of distance-regular graphs.

Recall that a subgraph induced on a subset Δ of X is said to be *strongly closed* if $C(u, v) \cup A(u, v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. Suzuki [18] determined all the types of strongly closed subgraphs of a distance-regular graph.

A distance-regular graph Γ with diameter d is said to be *d-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$.

Weng [23,24] used the term *weak-geodetically closed subgraphs* for strongly closed subgraphs, obtained the basic properties and characterized when a distance-regular graph is d-bounded. A regular strongly closed subgraph of Γ is said to be a *subspace* of Γ. For any two subspaces Δ_1 and Δ_2 of Γ, $\Delta_1 + \Delta_2$ denotes the minimum subspace containing Δ_1 and Δ_2.

The lattices generated by orbits of subspaces under finite classical groups were discussed in Huo, Liu and Wan [13–15], Huo and Wan [16], Gao and You [6,7], Wang and Feng [19], Wang and Guo [20,21], Guo and Nan [12,17], Guo [10], Guo, Li and Wang [11], Wang and Li [22]. The subspaces of a d-bounded distance-regular graph have similar properties to those of a vector space, as a generalization of the above results, some lattices were constructed by subspaces in a d-bounded distance-regular graph, see [4,8,9].

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \ge 3$. For $x \in X$, let $P(x)$ be the set of all subspaces containing x in Γ. For $1 \le i \le d-1$, let Δ_0 denote a fixed subspace with diameter $d - i$ in $P(x)$, and let

$$\mathscr{L}(d, i) = \{\Delta \in P(x) \mid \Delta + \Delta_0 = \Gamma, d(\Delta) = d(\Delta \cap \Delta_0) + i\} \cup \{\emptyset\}.$$

If we define the partial order on $\mathscr{L}(d, i)$ by ordinary inclusion (resp. reverse inclusion), then $\mathscr{L}(d, i)$ is a finite poset, denoted by $\mathscr{L}_O(d, i)$ (resp. $\mathscr{L}_R(d, i)$). In the present paper we show that both $\mathscr{L}_O(d, i)$ and $\mathscr{L}_R(d, i)$ are atomic, and compute their characteristic polynomials.

2. Some results on *d*-bounded distance-regular graphs

In this section we first recall some results on d-bounded distance-regular graphs, and then introduce two useful lemmas.

Proposition 2.1 (*[23, Lemma 2.6]*). *Let Γ be a d-bounded distance-regular graph. Then we have $b_i > b_{i+1}$ where $0 \le i \le d-1$.*

Proposition 2.2 (*[24, Lemmas 4.2, 4.5]*). *Let Γ be a d-bounded distance-regular graph. Then the following hold:*

(i) *Let Δ be a subspace of Γ and $0 \le i \le d(\Delta)$. Then Δ is distance-regular with intersection numbers $c_i(\Delta) = c_i$, $a_i(\Delta) = a_i$, $b_i(\Delta) = b_i - b_{d(\Delta)}$.*
(ii) *For any vertices x and y, the subspace with diameter $\partial(x, y)$ containing x, y is unique.*

Proposition 2.3 (*[4, Lemma 2.1]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \ge 2$. For $1 \le i+1 \le i+s \le i+s+t \le d$, suppose that Δ and Δ' are two subspaces satisfying $\Delta \subseteq \Delta'$, $d(\Delta) = i$ and $d(\Delta') = i+s+t$. Then the number of the subspaces with diameter $i+s$ containing Δ and contained in Δ', denoted by $N(i, i+s, i+s+t)$, is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}.$$

Proposition 2.4 (*[4, Lemma 2.8]*). *Let Γ be a d-bounded distance-regular graph. Suppose that Δ and Δ' are two subspaces. If $d(\Delta \cap \Delta') \ne \emptyset$, then $d(\Delta) + d(\Delta') \ge d(\Delta \cap \Delta') + d(\Delta + \Delta')$.*

Proposition 2.5 (*[5, Lemma 2.8]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \ge 2$. Let Δ and Δ' be two subspaces in Γ such that $d(\Delta) + d(\Delta') = d(\Delta \cap \Delta') + d(\Delta + \Delta')$. Then for all subspaces Δ_1 containing $\Delta \cap \Delta'$ in Δ, and for all subspaces Δ_2 containing $\Delta \cap \Delta'$ in Δ', we have $d(\Delta_1) + d(\Delta_2) = d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2)$.*

Proposition 2.6 (*[5, Theorem 1.1]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \ge 3$. Suppose that $0 \le t \le i+t, j+t \le i+j+t \le d_1 \le d$, and suppose that Δ and Δ^* are subspaces with diameter $i+t$ and diameter d_1 in $P(x)$, respectively. Suppose $\Delta \subseteq \Delta^*$. Then the number of subspaces Δ' with diameter $j+t$ and $\Delta' \subseteq \Delta^*$ in $P(x)$ such that $d(\Delta \cap \Delta') = t$ and $d(\Delta + \Delta') = i+j+t$, denoted by $M(t, i+t, j+t; d_1)$, is*

$$\frac{(b_0 - b_{i+t})(b_1 - b_{i+t}) \cdots (b_{t-1} - b_{i+t})(b_{i+t} - b_{d_1})(b_{i+t+1} - b_{d_1}) \cdots (b_{i+j+t-1} - b_{d_1})}{(b_0 - b_t)(b_1 - b_t) \cdots (b_{t-1} - b_t)(b_t - b_{j+t})(b_{t+1} - b_{j+t}) \cdots (b_{j+t-1} - b_{j+t})}.$$

In the rest of the paper, we always assume that $M(l, d-i, i+l; d)$ is given by Proposition 2.6. Now we prove two basic results on $\mathscr{L}(d, i)$.

Lemma 2.7. *Let $\Delta \in \mathscr{L}(d, i) \setminus \{\emptyset\}$ and $\Delta \supseteq \Delta_1 \in P(x)$. Then $\Delta_1 \in \mathscr{L}(d, i)$ if and only if $(\Delta \cap \Delta_0) + \Delta_1 = \Delta$ and $d(\Delta_1) = d(\Delta_1 \cap \Delta_0) + i$.*