Discrete Mathematics 310 (2010) 767–773

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Some new results on walk regular graphs which are cospectral to its complement

Mirko Lepović

Tihomira Vuksanovića 32, 34000, Kragujevac, Serbia

ARTICLE INFO

Article history:
Received 2 September 2008
Received in revised form 1 September 2009
Accepted 8 September 2009
Available online 26 September 2009

Keywords:
Walk regular graph
Strongly regular graph
Vertex-deleted subgraph
Eigenvalue
Main eigenvalue

ABSTRACT

We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. We say that a graph G of order n is walk regular if and only if its vertex deleted subgraphs $G_i = G \setminus i$ are cospectral for $i = 1, 2, \ldots, n$. Let G be a walk regular graph of order $4k + 1$ and degree $2k$ which is cospectral to its complement $\overline{G}$. Let H_i be switching equivalent to G_i with respect to $S_i \subseteq V(G_i)$. We here prove that G is strongly regular if and only if $\varDelta(G_i) = \varDelta(H_i)$ for $i = 1, 2, \ldots, 4k + 1$, where $\varDelta(G)$ is the number of triangles of a graph G.

1. Introduction

Let G be a simple graph of order n. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively. Let

$$P_G(\lambda) = \sum_{k=0}^{n} a_k \lambda^{n-k} \quad \text{and} \quad P_{\overline{G}}(\lambda) = \sum_{k=0}^{n} \overline{a}_k \lambda^{n-k},$$

where $\overline{G}$ denotes the complement of G. We know that $a_0 = 1$, $a_1 = 0$, $a_2 = -e$ and $a_3 = -2\varDelta$, where $e = e(G)$ and $\varDelta = \varDelta(G)$ is the number of edges and the number of triangles of the graph G.

Let $A^k = [a_{ij}^{(k)}]$ for any non-negative integer k. The number W_k of all walks of length k in G equals $\mathbf{sum}\, A^k$, where $\mathbf{sum}\, M$ is the sum of all elements in a matrix M. According to [1], the generating function $W_G(t)$ of the numbers W_k of length k in the graph G is defined by $W_G(t) = \sum_{k=0}^{+\infty} W_k t^k$. Besides [1],

$$W_G(t) = \frac{1}{t}\left[\frac{(-1)^n P_{\overline{G}}\left(-\frac{t+1}{t}\right)}{P_G\left(\frac{1}{t}\right)} - 1\right]. \tag{1}$$

Similarly, the function $W_G^*(t) = \sum_{k=0}^{+\infty} W_k^* t^k$ is called the Seidel generating function [5], where $W_k^* = \mathbf{sum}\,(A^*)^k$. Further, we say that an eigenvalue μ of G is main if and only if $\langle \mathbf{j}, \mathbf{Pj} \rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates

E-mail address: lepovic@knez.uis.kg.ac.yu.

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doi:10.1016/j.disc.2009.09.011

equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. Since $W_k = \langle A^m\mathbf{j}, A^{k-m}\mathbf{j}\rangle$ we find that

$$W_k = W_{1,m}W_{1,k-m} + W_{2,m}W_{2,k-m} + \cdots + W_{n,m}W_{n,k-m}, \tag{2}$$

where $W_{i,k}$ is the number of all walks of length k that starts from the vertex i. In particular, using (2) we obtain (i) $W_0 = n$; (ii) $W_1 = \sum_{i=1}^n d_i$; (iii) $W_2 = \sum_{i=1}^n d_i^2$ and (iv) $W_3 = \sum_{(i,j)\in E} d_i d_j$ where $d_i = d_i(G)$ denotes the degree of the vertex i in G and $E = E(G)$ is the edge set of G. Finally, using (1) we get

$$\sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}\overline{a}_i = a_k + \sum_{i=0}^{k-1} a_{k-1-i}W_i, \tag{3}$$

for any non-negative integer k. We say that $\mu^* \in \sigma^*(G)$ is the Seidel main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^*\mathbf{j}\rangle = n\cos^2\gamma > 0$, where $\mathbf{P}^*$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_{A^*}(\mu^*)$. The quantity $\beta^* = |\cos\gamma|$ is called the Seidel main angle of μ^*. Using the spectral decomposition of A^*, we find that

$$W_G^*\left(\frac{1}{\lambda}\right) = \frac{n_1^*\lambda}{\lambda-\mu_1^*} + \frac{n_2^*\lambda}{\lambda-\mu_2^*} + \cdots + \frac{n_k^*\lambda}{\lambda-\mu_k^*}, \tag{4}$$

where $n_i^* = n(\beta_i^*)^2$ and $n_1^* + n_2^* + \cdots + n_k^* = n$, understanding that β_i^* is the Seidel main angle of μ_i^*. Of course, for any $\lambda^* \in \sigma^*(G)$ we have $-\lambda^* \in \sigma^*(\overline{G})$. Since $\mathcal{E}_{A^*}(\lambda^*) = \mathcal{E}_{\overline{A}^*}(-\lambda^*)$, we obtain that $\mathcal{M}^*(\overline{G}) = -\mathcal{M}^*(G)$, where $-\mathcal{M}^*(G) = \{\lambda^* \mid -\lambda^* \in \mathcal{M}^*(G)\}$. Therefore, according to (4), we get

$$W_{\overline{G}}^*\left(\frac{1}{\lambda}\right) = \frac{n_1^*\lambda}{\lambda+\mu_1^*} + \frac{n_2^*\lambda}{\lambda+\mu_2^*} + \cdots + \frac{n_k^*\lambda}{\lambda+\mu_k^*}. \tag{5}$$

Next, the Seidel spectrum of a graph G which is cospectral to its complement $\overline{G}$ is symmetric with respect to the zero point. Since, in this case, $W_G^*(t) = W_{\overline{G}}^*(t)$ it follows that the Seidel main spectrum of G is also symmetric with respect to the zero point. Consequently, according to (4) and (5), we note if $\mu_+^*, \mu_-^* \in \mathcal{M}^*(G)$ then $n_+^* = n_-^*$, where $n_+^* = n(\beta_+^*)^2$ and $n_-^* = n(\beta_-^*)^2$ are related to μ_+^* and $\mu_-^* = -\mu_+^*$, respectively.

Remark 1. Let $\mathcal{M}(G)$ be the set of all main eigenvalues of G. Then we have $|\mathcal{M}(G)| = |\mathcal{M}(\overline{G})|$ and $|\mathcal{M}(G)| = |\mathcal{M}^*(G)|$, where $\mathcal{M}^*(G)$ denotes the set of all Seidel main eigenvalues of G.

2. Some preliminary results

Let i be a fixed vertex from the vertex set $V(G) = \{1, 2, \ldots, n\}$ and let $G_i = G \setminus i$ be its corresponding vertex deleted subgraph. Let S_i denote the neighborhood of i, defined as the set of all vertices of G which are adjacent to i. Besides, let $\Delta_i = \sum_{j\in S_i} d_j$.

Proposition 1 (*Lepović [7]*). *Let G be a connected or disconnected graph of order n. Then, for any vertex deleted subgraph G_i we have:*

(1^0) $\Delta_j(G_i) = \Delta_j(G) - d_i(G) - a_{ij}^{(2)}(G)$ *if* $j \in S_i$;

(2^0) $\Delta_j(G_i) = \Delta_j(G) - a_{ij}^{(2)}(G)$

if $j \in T_i = V(G_i) \setminus S_i$.

We say that a regular graph G of order n and degree $r \geq 1$ is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, understanding that G is not the complete graph K_n. We know that a regular connected graph is strongly regular if and only if it has exactly three distinct eigenvalues [2].

Theorem 1 (*Lepović [7]*). *A regular graph G of order n and degree $r \geq 1$ is strongly regular if and only if its vertex deleted subgraphs G_i have exactly two main eigenvalues for $i = 1, 2, \ldots, n$.*

Theorem 2 (*Lepović [7]*). *Let G be a connected or disconnected strongly regular graph of order n and degree r. Then for any vertex deleted subgraph G_i we have*

$$\mu_{1,2} = \frac{\tau - \theta + r \pm \sqrt{(\tau-\theta-r)^2 - 4\theta}}{2},$$

where μ_1 and μ_2 are the main eigenvalues of G_i.