Discrete Mathematics 310 (2010) 782–791

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Scattering matrices of regular coverings of graphs

Hirobumi Mizuno [a], Iwao Sato [b,*]

[a] *Iond University, Tokyo, Japan*

[b] *Oyama National College of Technology, Oyama, Tochigi 323–0806, Japan*

ARTICLE INFO

Article history:
Received 24 May 2008
Received in revised form 3 September 2009
Accepted 8 September 2009
Available online 24 September 2009

Keywords:
Laplacian matrix
Scattering matrix
Graph covering
L-function

ABSTRACT

We give a decomposition formula for the determinant on the bond scattering matrix of a regular covering of G. Furthermore, we define an L-function of G, and give a determinant expression of it. As a corollary, we express the determinant on the bond scattering matrix of a regular covering of G by means of its L-functions.

1. Introduction

Graphs treated here are finite. Let G be a connected graph (possibly with multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges. We write uv for an edge joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* arc of $e = (u, v)$.

A *path P of length n* in G is a sequence $P : (v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$ of $n + 1$ vertices and n arcs such that $v_0 \in V(G)$, $v_i \in V(G)$, $e_i \in D(G)$ and $e_i = (v_{i-1}, v_i)$ for $1 \le i \le n$. We write $P = (e_1, \ldots, e_n)$. Set $|P| = n$, $o(P) = v_0$ and $t(P) = v_n$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \ldots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \ldots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \ldots, e_1^{-1})$. As standard terminologies of graph theory, a path and a cycle are a diwalk and a closed diwalk, respectively.

We introduce an equivalence relation on the set of cycles. Two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class that contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara zeta function* of a graph G is defined to be a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small, by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G (see [9]).

* Corresponding author. Tel.: +81 285 20 2176; fax: +81 285 20 2880.
E-mail address: isato@oyama-ct.ac.jp (I. Sato).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.09.009

Zeta functions of graphs were originally defined for regular graphs by Ihara [9]. In [9], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [15,16]. Hashimoto [8] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph and showed that its reciprocal is again a polynomial.

Theorem 1 (*Bass*). *If G is a connected graph, then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}(G,u)^{-1} = (1-u^2)^{r-1}\det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D}-\mathbf{I})),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = \mathbf{D}_G = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ where $V(G) = \{v_1, \ldots, v_p\}$.

Stark and Terras [14] gave an elementary proof of Theorem 1 and discussed three different zeta functions of any graph. Other proofs of Bass' Theorem were given by Foata and Zeilberger [5] and Kotani and Sunada [10].

The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph (see [3,4,7,13]).

Smilansky [13] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs and expressed the determinant on the bond scattering matrix of a graph G by using the characteristic polynomial of its Laplacian.

Let G be a connected graph with $V(G) = \{v_1, \ldots, v_p\}$ and $D(G) = \{e_1, \ldots, e_q, e_1^{-1}, \ldots, e_q^{-1}\}$. The *Laplacian (matrix)* $\mathbf{L} = \mathbf{L}(G)$ of G is defined by

$$\mathbf{L} = -\mathbf{A}(G) + \mathbf{D}.$$

Let λ be a eigenvalue of $\mathbf{L}$, and let $\phi = (\phi_1, \ldots, \phi_p)$ be the eigenvector corresponding to λ. For each arc $b = (v_j, v_l)$, one associates a *bond wave function*

$$\phi_b(x) = a_b e^{i\pi x/4} + a_{b^{-1}} e^{-i\pi x/4}, \quad x = \pm 1, i = \sqrt{-1}$$

under the condition

$$\phi_b(1) = \phi_j \quad \text{and} \quad \phi_b(-1) = \phi_l.$$

We consider the following three conditions:

1. *uniqueness*: The value ϕ_j of the eigenvector at the vertex v_j computed in the terms of the bond wave functions, is the same for all the arcs emanating from v_j.
2. *ϕ is an eigenvector of* $\mathbf{L}$;
3. *consistency*: The linear relation between the incoming and the outgoing coefficients must be satisfied simultaneously at all vertices.

By uniqueness, we have

$$a_{b_1} e^{i\pi/4} + a_{b_1^{-1}} e^{-i\pi/4} = a_{b_2} e^{i\pi/4} + a_{b_2^{-1}} e^{-i\pi/4} = \cdots = a_{b_{d_j}} e^{i\pi/4} + a_{b_{d_j}^{-1}} e^{-i\pi/4},$$

where $b_1, b_2, \ldots, b_{d_j}$ are arcs emanating from v_j, and $d_j = \deg v_j$.

By condition 2, we have

$$-\sum_{k=1}^{d_j} (a_{b_k} e^{-i\pi/4} + a_{b_k^{-1}} e^{i\pi/4}) = (\lambda - d_j)\frac{1}{d_j}\sum_{k=1}^{d_j}(a_{b_k} e^{i\pi/4} + a_{b_k^{-1}} e^{-i\pi/4}).$$

Thus, for each arc b with $o(b) = v_j$,

$$a_b = \sum_{t(c)=v_j} \sigma_{bc}^{v_j}(\lambda) a_c, \tag{1}$$

where

$$\sigma_{bc}^{v_j}(\lambda) = i\left(\delta_{b^{-1}c} - \frac{2}{d_j}\frac{1}{1 - i(1-\lambda/d_j)}\right)$$

and $\delta_{b^{-1}c}$ is the Kronecker delta. The *bond scattering matrix* $\mathbf{U}(\lambda) = (U_{ef})_{e,f \in D(G)}$ of G is defined by

$$U_{ef} = \begin{cases} \sigma_{ef}^{t(f)} & \text{if } t(f) = o(e), \\ 0 & \text{otherwise.} \end{cases}$$

By consistency, we have

$$\mathbf{U}(\lambda)\mathbf{a} = \mathbf{a},$$

where $\mathbf{a} = {}^t(a_1, a_2, \ldots, a_{2q})$. This holds if and only if

$$\det(\mathbf{I}_{2q} - \mathbf{U}(\lambda)) = 0.$$