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The asymptotic number of spanning trees in circulant graphs[☆]

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ABSTRACT

Let $T(G)$ be the number of spanning trees in graph G. In this note, we explore the asymptotics of $T(G)$ when G is a circulant graph with given jumps.

The circulant graph $C_n^{s_1,s_2,\dots,s_k}$ is the $2k$-regular graph with n vertices labeled $0, 1, 2, \dots, n-1$, where node i has the $2k$ neighbors $i \pm s_1, i \pm s_2, \dots, i \pm s_k$ where all the operations are $(\bmod n)$. We give a closed formula for the asymptotic limit $\lim_{n\to\infty} T(C_n^{s_1,s_2,\dots,s_k})^{\frac{1}{n}}$ as a function of $s_1, s_2, \dots, s_k$. We then extend this by permitting some of the jumps to be linear functions of n, i.e., letting s_i, d_i and e_i be arbitrary integers, and examining

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\dots,s_k,\lfloor\frac{n}{d_1}\rfloor+e_1,\lfloor\frac{n}{d_2}\rfloor+e_2,\dots,\lfloor\frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}}.$$

While this limit does not usually exist, we show that there is some p such that for $0 \le q < p$, there exists c_q such that limit (1) restricted to only n congruent to q modulo p does exist and is equal to c_q. We also give a closed formula for c_q.

One further consequence of our derivation is that if s_i go to infinity (in any arbitrary order), then

$$\lim_{s_1,s_2,\dots,s_k\to\infty}\ \lim_{n\to\infty} T(C_n^{s_1,s_2,\dots,s_k})^{\frac{1}{n}} = 4\exp\left[\int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2\pi x_i\right)dx_1dx_2\cdots dx_k\right].$$

Interestingly, this value is the same as the asymptotic number of spanning trees in the k-dimensional square lattice recently obtained by Garcia, Noy and Tejel.

1. Introduction

Throughout this paper, we permit graphs (digraphs) to contain multiple edges (arcs) and self-loops unless otherwise specified. Let G and D denote a graph and a digraph, respectively. *A spanning tree* in G is a tree having the same vertex set as G. *An oriented spanning tree* in D is a rooted tree with the same vertex set as D, i.e., there is a specified root node and paths from it to every vertex of D. The study of the number of spanning trees in a graph has a long history. Evaluating this number is

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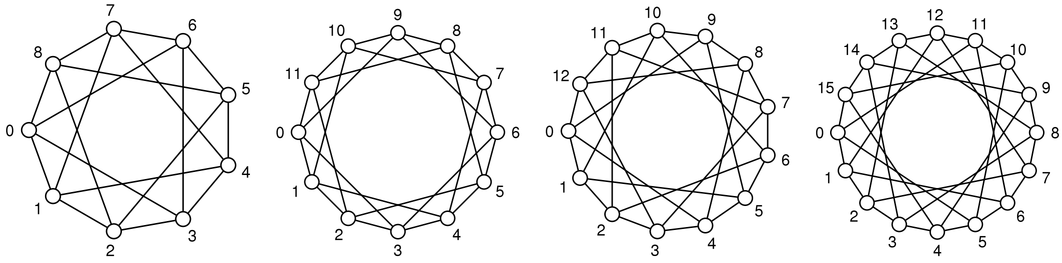

(a) $C_9^{1,3}$.

(b) $C_{12}^{1,3}$.

(c) $C_{13}^{1,4}$.

(d) $C_{16}^{1,5}$.

Fig. 1. 4 circulant graphs. (b) and (d) are $C_{4n+8}^{1,2n+1}$ for $n = 1, 2$. (a) and (c) are $C_{4n+1}^{1,n+1}$ for $n = 2, 3$.

not only interesting from a combinatorial perspective but also arises in practical applications, e.g., analyzing the reliability of a network in the presence of line faults, designing electrical circuits etc. [8]. Given the adjacency matrix of G (or D), *Kirchoff's matrix tree theorem* [12] gives a closed formula for calculating the number of spanning trees. The real problem, then, is to calculate the number of spanning trees of graphs in particular parameterized classes, as a function of the parameters. A well studied class, which we will be further analyzing in this paper, is the *circulant graphs*.

We start by formally defining the graphs and the values to be counted. Let $s_1, s_2, \ldots, s_k$ be positive integers. The circulant graph with n vertices and jumps $s_1, s_2, \ldots, s_k$ is defined by

$$C_n^{s_1,s_2,\ldots,s_k} = (V, E)$$

where

$$V = \{0, 1, 2, \ldots, n-1\}, \quad \text{and} \quad E = \bigcup_{i=0}^{n-1} \{(i, i \pm s_1), (i, i \pm s_2), \ldots, (i, i \pm s_k)\}$$

where all of the additions are modulo n. That is, each node is connected to the nodes that are jumps $\pm s_j$ away from it, for $j = 1, 2, \ldots, k$.[1] Similarly the directed circulant graph, $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, has the same vertex set, but

$$E = \bigcup_{i=0}^{n-1} \{(i, i + s_1), (i, i + s_2), \ldots, (i, i + s_k)\}$$

i.e, there is an edge directed from each i to the nodes s_j ahead of it, for $j = 1, 2, \ldots, k$. Examples of four undirected circulant graphs are given in Fig. 1.

We will use $T(X)$ to denote the number of spanning trees in a directed or undirected graph X. It was shown in [17] that, for directed circulant graphs,

$$\lim_{n\to\infty} \frac{T(\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k})}{T(\vec{C}_n^{s_1,s_2,\ldots,s_k})} = k,$$

where k is the degree of each vertex of $\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k}$. One might hope that similar asymptotic behavior, i.e., a limit dependent only upon k but independent of the actual values of the s_i, would also be true for undirected circulant graphs. Unfortunately, as seen in the asymptotic (numerical) results presented in Table 1 of [18], this is not the case; the asymptotic limits do seem somehow dependent on the s_i.

We therefore, in that paper, posed "*the analysis of the asymptotics as a function of the s_i*" as an open question. This paper addresses that question.

The problem of calculating the asymptotic maximum number of spanning trees in a circulant graph with k jumps was treated in [13], but their technique does not seem to permit analyzing the number of spanning trees for any given fixed jumps. Asymptotic limits for grids and tori (which turn out to be equal) were obtained in [6,9]. More recently, while examining the structure of non-constant jump circulant graphs, it was conjectured in [10] that the asymptotics of the number of spanning trees of the $m \times n$ tori and grids and the circulant graphs $C_{mn}^{1,n}$ would be the same.

The main result of this paper is the derivation in Section 2 of closed formulas for the first order asymptotics of the number of spanning trees in undirected circulant graphs, both for fixed jump circulants and linear jump ones (in which the jump

[1] To avoid confusion, we emphasize that, since we are allowing multiple edges in our graphs, $C_n^{s_1,s_2,\ldots,s_k}$ is *always* $2k$-regular and $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is *always* k-regular. For example, in our notation, $C_{2n}^{1,n}$ is the 4-regular graph with $2n$ vertices such that each vertex i is connected by *one* edge to each of $(i-1) \bmod 2n$ and $(i+1) \bmod 2n$ and by *two* edges to $(i+n) \bmod 2n$. Our techniques would, with slight technical modifications, also permit analyzing graphs in which multiple edges are *not* allowed, e.g., the *Mobius ladder* M_{2n}. This is the 3-regular graph with $2n$ vertices such that each vertex i is connected by one edge to each of $(i-1) \bmod 2n$, $(i+1) \bmod 2n$ and $(i+n) \bmod 2n$. The reason that we do not explicitly analyze such graphs is that such an analysis would require rewriting all of our theorems a second time to deal with these special instances without introducing any new interesting techniques.