Discrete Mathematics 310 (2010) 832–837

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Some remarks on the geodetic number of a graph

Mitre C. Dourado [a], Fábio Protti [b], Dieter Rautenbach [c], Jayme L. Szwarcfiter [d]

[a] ICE, Universidade Federal Rural do Rio de Janeiro and NCE - UFRJ, Brazil

[b] Universidade Federal Fluminense, Instituto de Computação, Niterói, RJ, Brazil

[c] Institut für Mathematik, Technische Universität Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany

[d] Universidade Federal do Rio de Janeiro, Instituto de Matemática, NCE and COPPE, Caixa Postal 2324, 20001-970 Rio de Janeiro, RJ, Brazil

ARTICLE INFO

Article history:
Received 17 March 2009
Received in revised form 17 September 2009
Accepted 19 September 2009
Available online 2 October 2009

Keywords:
Cograph
Convex hull
Convex set
Geodetic number
Split graph
Unit interval graph

ABSTRACT

A set of vertices D of a graph G is geodetic if every vertex of G lies on a shortest path between two not necessarily distinct vertices in D. The geodetic number of G is the minimum cardinality of a geodetic set of G.

We prove that it is NP-complete to decide for a given chordal or chordal bipartite graph G and a given integer k whether G has a geodetic set of cardinality at most k. Furthermore, we prove an upper bound on the geodetic number of graphs without short cycles and study the geodetic number of cographs, split graphs, and unit interval graphs.

1. Introduction

We consider finite, undirected and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The neighbourhood of a vertex u in G is denoted by $N_G(u)$. A set of pairwise non-adjacent vertices is called independent and a set of pairwise adjacent vertices is called a clique. A vertex is simplicial if its neighbourhood is a clique. The distance $d_G(u, v)$ between two vertices u and v in G is the length of a shortest path between u and v or ∞, if no such path exists. The diameter of G is the maximum distance between two vertices in G.

The *interval* $I[u, v]$ between two vertices u and v in G is the set of vertices of G which belong to a shortest path between u and v. Note that a vertex w belongs to $I[u, v]$ if and only if $d_G(u, v) = d_G(u, w) + d_G(w, v)$. For a set S of vertices, let the *interval* $I[S]$ of S be the union of the intervals $I[u, v]$ over all pairs of vertices u and v in S. A set of vertices S is called *geodetic* if $I[S]$ contains all vertices of G. Harary et al. [12] defined the *geodetic number* $g(G)$ of a graph G as the minimum cardinality of a geodetic set. The calculation of the geodetic number is an NP-hard problem for general graphs [3] and [2,7–10,13] contain numerous results and references concerning geodetic sets and the geodetic number.

Our results are as follows. In Section 2 we simplify and refine the existing complexity result [3] by proving that the decision problem corresponding to the geodetic number remains NP-complete even when restricted to chordal or chordal bipartite graphs. In Section 3 we prove upper bounds on the geodetic number of graphs without short cycles and in particular for triangle-free graphs. Finally, in Section 4 we consider the geodetic number of cographs, split graphs and unit interval graphs.

E-mail addresses: mitre@nce.ufrj.br (M.C. Dourado), fabio@ic.uff.br (F. Protti), dieter.rautenbach@tu-ilmenau.de (D. Rautenbach), jayme@nce.ufrj.br (J.L. Szwarcfiter).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.09.018

2. Complexity results for chordal graphs

In this section we prove hardness results for the following decision problem.

Geodetic Set

Instance: A graph G and an integer k.
Question: Does G have a geodetic set of cardinality at most k?

Our proofs will relate Geodetic Set to the following well-known problem. Recall that a set of vertices D of a graph G is *dominating* if every vertex in $V(G) \setminus D$ has a neighbour in D.

Dominating Set

Instance: A graph G and an integer k.
Question: Does G have a dominating set of cardinality at most k?

A graph is *chordal* if it does not contain an induced cycle of length at least 4. Similarly, a bipartite graph is *chordal bipartite* if it does not contain an induced cycle of length at least 6. The problem Dominating Set is NP-complete for chordal graphs [5] and chordal bipartite graphs [16].

Theorem 1. Geodetic Set *restricted to chordal graphs is NP-complete.*

Proof. Since the interval of a given set of vertices can be determined in polynomial time by shortest path methods, Geodetic Set is in NP.

In order to prove NP-completeness, we describe a polynomial reduction of Dominating Set restricted to chordal graphs [5] to Geodetic Set restricted to chordal graphs. Let (G, k) be an instance of Dominating Set such that G is chordal. Let the graph G' arise from G as follows: For every vertex $u \in V(G)$, add two new vertices x_u and y_u and add the new edges ux_u and x_uy_u. Furthermore add a new vertex z and new edges uz and x_uz for every $u \in V(G)$. Let $k' = k + |V(G)|$. Note that G' is chordal.

If G has a dominating set D with $|D| \leq k$, then let $D' = D \cup \{y_u \mid u \in V(G)\}$. Clearly, $\{x_u \mid u \in V(G)\} \cup \{z\} \subseteq I[\{y_u \mid u \in V(G)\}] \subseteq I[D']$. Furthermore, if $u \in V(G) \setminus D$, then there is a vertex $v \in D$ with $uv \in E(G)$. Since $d_{G'}(v, y_u) = 3$ and vux_uy_u is a path of length 3 in G', we have $u \in I[v, y_u] \subseteq I[D']$. Hence D' is a geodetic set of G' with $|D'| \leq k + |V(G)| = k'$.

Conversely, if G' has a geodetic set D' with $|D'| \leq k'$, then let $D = D' \cap V(G)$. Clearly, D is not empty. Since $\{y_u \mid u \in V(G)\} \subseteq D'$, we have $|D| \leq k' - |V(G)| = k$. If $u \in V(G) \setminus D$, then either there are two vertices $v, w \in D$ with $u \in I[v, w]$ or there are two vertices $v \in D$ and $w \in D' \setminus V(G)$ with $u \in I[v, w]$. In both cases, the distances within G' imply that v must be a neighbour of u. Hence D is a dominating set of G with $|D| \leq k$. $\square$

Theorem 2. Geodetic Set *restricted to chordal bipartite graphs is NP-complete.*

Proof. In order to prove NP-completeness, we describe a polynomial reduction of Dominating Set restricted to chordal bipartite graphs [16] to Geodetic Set restricted to chordal bipartite graphs. Let (G, k) be an instance of Dominating Set such that G is chordal bipartite.

Let the graph G' arise from G as follows: Let A and B denote the partite sets of G. Add four new vertices a_1, a_2, b_1, b_2 and add new edges a_1b for all $b \in B \cup \{b_1, b_2\}$ and b_1a for all $a \in A \cup \{a_1, a_2\}$. Let $k' = k + 2$. Note that G' is chordal bipartite.

If G has a dominating set D with $|D| \leq k$, then let $D' = D \cup \{a_2, b_2\}$. Clearly, $a_1, b_1 \in I[a_2, b_2]$. Furthermore, if $a \in A \setminus D$, then there is a vertex $b \in D \cap B$ with $ab \in E(G)$. Since $d_{G'}(a_2, b) = 3$ and a_2b_1ab is a path of length 3 in G', we have $a \in I[a_2, b] \subseteq I[D']$. Hence, by symmetry, D' is a geodetic set of G' with $|D'| \leq k + 2 = k'$.

Conversely, if G' has a geodetic set D' with $|D'| \leq k'$, then let $D = D' \cap V(G)$. Clearly, $a_2, b_2 \in D'$ and D is not empty. If $a \in A \setminus D$, then either there are two vertices $b \in D \cap B$ and $v \in D$ with $a \in I[b, v]$ or there is a vertex $b \in D \cap B$ with $a \in I[b, a_2]$. In both cases, the distances within G' imply that a must be a neighbour of b. Hence, by symmetry, D is a dominating set of G with $|D| \leq k$. $\square$

3. Bounds for triangle-free graphs

In this section we prove upper bounds on the geodetic number for graphs without short cycles. The *girth* of a graph G is the length of a shortest cycle in G or ∞, if G has no cycles. Our first result is a probabilistic bound for graphs of large girth.

Theorem 3. *If G is a graph of order n, girth at least $4h$, and minimum degree at least δ, then*

(i)

$$g(G) \leq n\left(p + \delta(1-p)^{(\delta-1)\frac{(\delta-1)^h-1}{\delta-2}+1} - (\delta-1)(1-p)^{\delta\frac{(\delta-1)^h-1}{\delta-2}+1}\right)$$

for every $p \in (0, 1)$ and