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Hall–Littlewood polynomials and fixed point enumeration

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ABSTRACT

We resolve affirmatively some conjectures of Reiner, Stanton, and White (2004) [12] regarding enumeration of transportation matrices which are invariant under certain cyclic row and column rotations. Our results are phrased in terms of the bicyclic sieving phenomenon introduced by Barcelo, Reiner, and Stanton (2009) [1]. The proofs of our results use various tools from symmetric function theory such as the Stanton–White rim hook correspondence (Stanton and White (1985) [18]) and results concerning the specialization of Hall–Littlewood polynomials due to Lascoux, Leclerc, and Thibon (1994, 1997) [5,6].

1. Introduction and main results

Let X be a finite set and $C \times C'$ be a direct product of two finite cyclic groups acting on X. Fix generators c and c' for C and C' and let $\zeta, \zeta' \in \mathbb{C}$ be two roots of unity having the same multiplicative orders as c, c'. Let $X(q,t) \in \mathbb{C}[q,t]$ be a polynomial in two variables. Following Barcelo, Reiner, and Stanton [1], we say that the triple $(X, C \times C', X(q,t))$ *exhibits the bicyclic sieving phenomenon* (biCSP) if for any integers $d, e \geq 0$ the cardinality of the fixed point set $X^{(c^d, c'^e)}$ is equal to the polynomial evaluation $X(\zeta^d, \zeta^e)$. The biCSP encapsulates several combinatorial phenomena: specializing to the case where one of the cyclic groups is trivial yields the *cyclic sieving phenomenon* of Reiner, Stanton, and White [12] and specializing further to the case where the nontrivial cyclic group has order 2 yields the $q = -1$ *phenomenon* of Stembridge [19]. Moreover, the fact that the identity element in any group action fixes everything implies that whenever $(X, C \times C', X(q,t))$ exhibits the biCSP, we must have that the $q = t = 1$ specialization $X(1,1)$ is equal to the cardinality $|X|$ of the set X. In this paper we prove a pair of biCSPs conjectured by Reiner, Stanton, and White where the sets X are certain sets of matrices acted on by row and column rotation and the polynomials $X(q,t)$ are bivariate deformations of identities arising from the RSK insertion algorithm. Our proof, outlined in Section 2, relies on symmetric function theory and plethystic substitution. In Section 3 we outline an alternative argument due to Victor Reiner which proves these biCSPs 'up to modulus' using DeConcini–Procesi modules.

Given a partition $\lambda \vdash n$, recall that a *semistandard Young tableau (SSYT) of shape* λ is a filling of the Ferrers diagram of λ with positive numbers which increase strictly down columns and weakly across rows. For a SSYT T of shape λ, the *content* of T is the (weak) composition $\mu \models n$ given by letting μ_i equal the number of i's in T. A SSYT T is called *standard* (SYT) if it has content 1^n. For a partition λ and a composition μ of n, the *Kostka number* $K_{\lambda,\mu}$ is equal to the number of SSYT of shape λ and content μ.

The *Kostka–Foulkes polynomials* $K_{\lambda,\mu}(q)$, indexed by a partition $\lambda \vdash n$ and a composition $\mu \models n$, arose originally as the entries of the transition matrix between the Schur function and Hall–Littlewood symmetric function bases of the ring of

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symmetric functions (with coefficients in $\mathbb{C}(q)$ where q is an indeterminate). A combinatorial proof of the positivity of their coefficients was given by Lascoux and Schützenberger [7] by identifying $K_{\lambda,\mu}(q)$ as the generating function for the statistic of *charge* on the set of semistandard tableaux of shape λ and content μ. We outline the definition of charge as the rank function of a cyclage poset.

Let $\mathcal{A}^*$ denote the free monoid of words $w_1 \ldots w_k$ of any length with letters drawn from $[n]$. Let $\equiv$ be the equivalence relation on $\mathcal{A}^*$ induced by

$$RkijR' \equiv RikjR', \qquad RjikR \equiv RjkiR', \qquad RjiiR' \equiv RijiR', \qquad RjijR' \equiv RjjiR',$$

where $1 \le i < j < k \le n$ and R and R' are any words in the monoid $\mathcal{A}^*$. The *Robinson–Schensted–Knuth correspondence* yields an algorithmic bijection between words w in $\mathcal{A}^*$ and pairs $(P(w), Q(w))$ of tableaux, where P is a SSYT with entries $\le n$ and Q is a SYT with the shape of $P(w)$ equal to the shape of $Q(w)$. For details on the RSK correspondence, see for example [14] or [17]. The RSK correspondence sets up an equivalence relation $\equiv'$ on words in $\mathcal{A}^*$ by setting $w \equiv' w'$ if and only if $Q(w) = Q(w')$. It is a result of Knuth [4] that the equivalence relations $\equiv$ and $\equiv'$ on $\mathcal{A}^*$ agree. That is, for any $w, w' \in \mathcal{A}^*$ we have $w \equiv w'$ if and only if $Q(w) = Q(w')$. Therefore, the quotient monoid $\mathcal{A}^*/\equiv$ is in a natural bijective correspondence with the set of SSYT with entries $\le n$. This quotient is called the *plactic monoid*.

Cyclage is a monoid analogue of the group operation of conjugation introduced by Lascoux and Schützenberger [8]. Given $w, w' \in \mathcal{A}^*/\equiv$, say that $w \prec w'$ if there exists $i \ge 2$ and $u \in \mathcal{A}^*/\equiv$ so that $w = iu$ and $w' = ui$. For a fixed composition $\mu \models n$, the transitive closure of the relation $\prec$ induces a partial order on the subset of $\mathcal{A}^*/\equiv$ consisting of words of content μ, and therefore also on the set of SSYT of content μ. For fixed μ, the rank generating function for this poset is called *cocharge* and is therefore a statistic on SSYT of content μ. The rank function of the order theoretic dual of this poset is called the *charge*. Lascoux and Scützenberger [7] proved that for any partition $\lambda \vdash n$ and any composition $\mu \models n$, we have that

$$K_{\lambda,\mu}(q) = \sum_T q^{charge(T)},$$

where the sum ranges over all SSYT T of shape λ and content μ.

For $n \ge 0$, define $\epsilon_n(q,t) \in \mathbb{N}[q,t]$ to be $(qt)^{n/2}$ if n is even and 1 if n is odd. The type A specialization of Theorem 1.4 of Barcelo, Reiner, and Stanton [1] yields the following:

Theorem 1.1 (*[1]*). *Let X be the set of $n \times n$ permutation matrices and $\mathbb{Z}_n \times \mathbb{Z}_n$ act on X by row and column rotation. The triple $(X, \mathbb{Z}_n \times \mathbb{Z}_n, X(q,t))$ exhibits the biCSP, where*

$$X(q,t) = \epsilon_n(q,t) \sum_{\lambda \vdash n} K_{\lambda,1^n}(q) K_{\lambda,1^n}(t).$$

Example 1.1. Let $n = 4$. We have that

$$X(q,t) = (qt)^2\Big[(qt)^6 + (qt)^3(1+q+q^2)(1+t+t^2) + (qt)^2(1+q^2)(1+t^2) + (qt)(1+q+q^2)(1+t+t^2) + 1\Big].$$

Consider the action of the diagonal subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$ on $X = S_4$. Let r be the generator of this subgroup, so that r acts on X by a simultaneous single row and column shift. We have that $X(i,i) = 4$, reflecting the fact that the fixed point set $X^r = \{1234, 2341, 3412, 4123\}$ has four elements. Also, $X(-1,-1) = 8$, whereas the fixed point set $X^{r^2} = \{1234, 2341, 3412, 4123, 1432, 2143, 3214, 4321\}$. Finally, we have that $X(i,-1) = 0$, reflecting the fact that no 4×4 permutation matrix is fixed by a simultaneous onefold row shift and twofold column shift.

The $q = t = 1$ specialization of $X(q,t)$ in the above result is implied by the RSK insertion algorithm on permutations. The following generalization of Theorem 1.1 to the case of words was known to Reiner and White but is unpublished. For any composition $\mu \models n$, let $\ell(\mu)$ denote the number of parts of μ and $|\mu| = n$ denote the sum of the parts of μ. A composition $\mu \models n$ is said to have cyclic symmetry of order a if one has $\mu_i = \mu_{i+a}$ always, where subscripts are interpreted modulo $\ell(\mu)$.

Theorem 1.2 (*[13,20]*). *Let $\mu \models n$ be a composition with cyclic symmetry of order $a|\ell(\mu)$. Let X be the set of length n words of content μ, thought of as 0, 1-matrices in the standard way. The product of cyclic groups $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_n$ acts on X by a-fold row rotation and onefold column rotation.*

The triple $(X, \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_n, X(q,t))$ exhibits the biCSP, where

$$X(q,t) = \epsilon_n(q,t) \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) K_{\lambda,1^n}(t).$$

Example 1.2. Let us give an example to show why the factor $\epsilon_n(q,t)$ is necessary in the definition of $X(q,t)$. Take $n = 2$, $\mu = (2)$, and $a = 1$. The set X is the singleton $\{11\}$ consisting of the word 11. One verifies that

$$K_{(1,1),(2)}(q) = 0, \qquad K_{(2),(2)}(q) = 1, \qquad K_{(1,1),(2)}(t) = 1, \qquad K_{(2),(2)}(t) = t,$$