Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3479–3490
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Path bundles on n-cubes

Matthew Elder

University of Wisconsin-Madison, 1210 W. Dayton St., Madison, WI 53706-1685, USA

Received 13 October 2006; received in revised form 18 June 2007; accepted 4 July 2007
Available online 10 September 2007

Abstract

A path bundle is a set of 2^a paths in an n-cube, denoted Q_n, such that every path has the same length, the paths partition the vertices of Q_n, the endpoints of the paths induce two subcubes of Q_n, and the endpoints of each path are complements. This paper shows that a path bundle exists if and only if $n > 0$ is odd and $0 \leqslant a \leqslant n - \lceil \log_2(n+1) \rceil$.

Keywords: Path bundle; n-Cube; Gray code; Binary words; Graph partition

1. Introduction

Path bundles are closely related to, and emerged from, the theory of binary Gray codes. Introduced as a coding mechanism in [1], binary Gray codes have proven useful in many various technical and mathematical applications. An overview of these applications, along with many generalizations and restrictions of binary Gray codes, is given in [3].

Path bundles are largely a generalization of binary Gray codes. An n-bit binary Gray code is a list of all binary words of length n where every pair of words adjacent in the list differ by only one bit. A path bundle is a set of 2^a paths in an n-cube such that every path has the same length, the paths partition the vertices of the n-cube, the endpoints of the paths induce two a-cubes of the n-cube, and the endpoints of each path are complements.

This paper shows that, for positive integers n and a, a path bundle of 2^a paths exists in an n-cube if and only if n is odd and $a \leqslant n - \lceil \log_2(n+1) \rceil$. The existence result is constructive; this paper gives an inductive construction of a path bundle for every possible n and a that satisfy these conditions.

Path bundles are useful for constructing binary Gray codes with various properties; they are the result of the author's search for a construction of a variant of the Gray codes in [2]. They encapsulate many details into a simple and predictable block; this may prove useful in other constructions of paths, circuits, or sets of paths on n-cubes.

2. Preliminaries

In Section 3 we provide conditions on path bundles necessary for their existence, and in Section 4 we show that these conditions are sufficient. First, we need some notation and definitions. We use the following definitions and notation

E-mail address: elder@cs.wisc.edu.

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.07.006

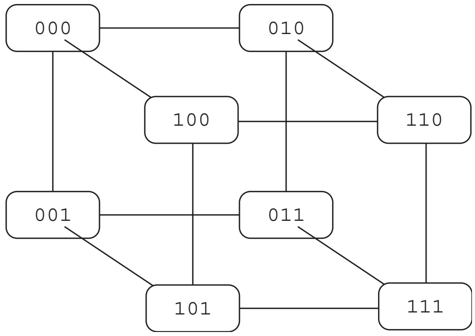

Fig. 1. The words of Q_3 and the 3-cube are isomorphic.

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

Fig. 2. A 4-cube (Q_4) labelled with the words it represents.

when dealing with binary words:

- *Bits* are binary digits, with possible values 0 and 1.
- *Words* are strings of bits; 0011, 0110, and 1001 are distinct words.
- The symbol $\oplus$ denotes addition in $\mathbb{Z}_2^n$ (the exclusive-or operation). For example, 000111 $\oplus$ 110011 = 110100.
- Denote a repeated bit or symbol with an exponent, like so: 00000 = 0^5, and 01^3 = 0111.
- If a and b are words, ab denotes the concatenation of a and b. So, if a = 11 and b = 0, then aab = 11110 and b^3a = 00011.
- Q_n is the set of all n-bit words.
- Two words w and v are *adjacent*, denoted $w \sim v$, if they differ by exactly one bit. So, 0101 and 1101 are adjacent. 0101 and 0110 are *not* adjacent.
- Two words are *complements* if they differ in every bit. The complement of a word a is denoted $\overline{a}$. For example, $\overline{0010}$ = 1101.
- Counting from 0, w_j is the jth bit of w. If w = 00100, then w_0 = 0, w_2 = 1, and w_5 is undefined.

Notice that Q_n and its adjacencies are isomorphic to the graph of an n-cube. Fig. 1 shows an example of this isomorphism. This isomorphism will not be useful for formal mathematics, but it guides both our intuition and our illustrations.

Rather than illustrating complex n-cubes in the manner of Fig. 1, we instead illustrate Q_n in the manner of Fig. 2. In this illustration, adjacent elements of Q_n are placed in corresponding positions in adjacent rectangles. We consider two rectangles adjacent if they share an edge unused by a larger, bolder rectangle.

For example, in Fig. 2, 0001 is adjacent to 0000 and 0101, because they are at (trivially) corresponding positions of adjacent rectangles in the larger, upper-left square. However, 0001 is not adjacent to 0010, because these small rectangles are not inside the same major subdivision; the edge they share is shared also by the upper pair of large rectangles. Also, 0001 is in the upper-right-hand corner of a 2×2 square, so it corresponds to 0011 and 1001.

This manner of illustration profits from the recursive nature of the n-cube; the n-cube is composed of two $n-1$-cubes whose corresponding vertices are connected. Crucially for our illustrations, an n-cube is also composed of four $n-2$-cubes, connected together in a square. Fig. 3 illustrates a 7-cube (Q_7) as composed of four 5-cubes (Q_5). We will illustrate cubes in this way throughout this paper.