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Graph-like spaces: An introduction

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ABSTRACT

Thomassen and Vella (Graph-like continua, augmenting arcs, and Menger's Theorem, Combinatorica, doi:10.1007/s00493-008-2342-9) have recently introduced the notion of a graph-like space, simultaneously generalizing infinite graphs and many of the compact spaces recently used by Diestel or Richter (and their coauthors) to study cycle spaces and related problems in infinite graphs. This work is a survey to introduce graph-like spaces and shows how many of these works on compact spaces can be generalized to compact graph-like spaces.

1. Introduction

In an on-going project, Diestel and his students have been studying compact topological spaces associated with certain infinite graphs, especially locally finite ones. This study has proved fruitful for generalizing many theorems about finite graphs to the infinite context; an essential point is that cycles are generalized to include infinite cycles. This has necessitated an inherently topological point of view: a cycle is now an embedding of a circle into the compact space. Particular examples of such works are [2–4,8].

The original motivation for some of these questions arose from Bonnington and Richter [1], who proved that the cycle space of a locally finite planar graph is generated by the face boundaries of a planar embedding, together with certain 2-way infinite paths joining the different accumulation points. However, the definition of cycle space here is different from that employed by Diestel. In the next section, we shall give a unified definition of cycle space; this section is devoted to providing historical context.

Motivated in part by a desire to unify the two definitions, Vella and Richter [23] introduced the notion of *edge space* and showed that the two notions of cycle space above are both instances of the cycle space of a compact edge space. The only difference is the embedding of the graph into a larger compact space. For locally finite graphs, Diestel uses the Freudenthal compactification, while Bonnington and Richter use the 1-point compactification.

However, edge spaces ignore many inherently graph-theoretic properties. Its name was chosen to reflect the emphasis on edges rather than vertices. Continuing this line of reasoning, Vella introduced the notion of *graph-like space* (appearing for the first time in [22]), which is the main focus of this introductory article.

A *graph-like space* is a metric space X with a 0-dimensional subspace V – the set of *vertices* – of X so that every component of $X - V$ is an open subset of X homeomorphic to an open arc and whose closure in X has only one or two additional points — these components of $X - V$ are the *edges*. (The meaning of 0-dimensional is: for any two points $u, w \in V$, there is a separation (U, W) of V so that $u \in U$ and $w \in W$. In particular, V is totally disconnected. If X is compact, then so is V and then totally disconnected is equivalent to being 0-dimensional.)

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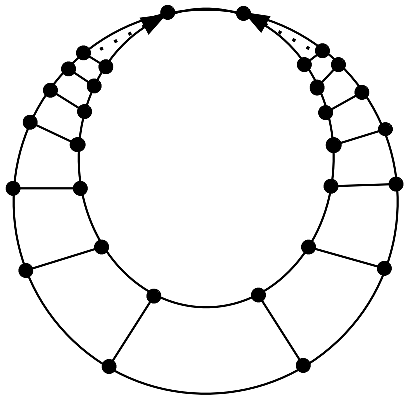

Fig. 1. 2-way infinite ladder plus an edge joining its ends.

From this definition, we see that any graph G is a graph-like space. (Give G the usual topology of a 1-dimensional cell complex; in this case, every subset of $V(G)$ is open in $V(G)$, so $V(G)$ has the discrete topology.) But many other topological spaces are graph-like: the Freudenthal and 1-point compactifications of an infinite, locally finite graph are also graph-like. Note that the added "points-at-infinity" are vertices of the graph-like space. We will be principally concerned with compact graph-like spaces.

In their more general results, Diestel et al. work with infinite graphs in which every two vertices are separated by some finite edge cut. For such a graph G, an identification space $\widetilde{G}$ obtained from the Freudenthal compactification of G is considered; it is known that $\widetilde{G}$ is another example of a compact graph-like space [23].

With the realization that the two notions of cycle space come from different compactifications of the same locally finite graph, suddenly a much broader landscape appears before us. Although the two compactifications considered above are the most natural, it is apparent that many others are possible. Moreover, one's horizons are no longer limited to compactifications of graphs. For example, in Fig. 1 we see the 2-way infinite ladder, plus its two ends, plus an edge joining the two ends. This is a compact graph-like space. Another compact graph-like space can be obtained from the 1-point compactification of the 2-way infinite path by joining the limit point to every vertex of the path.

The graph in the figure has many infinite cycles; for example, the two facial cycles bounding the infinite and the innermost faces both go through the edge containing the ends of the 2-way infinite ladder. Another feature of this space is that it is 3-connected (deleting any two vertices and their incident edges does not result in a disconnected subspace), whereas the 2-way infinite ladder by itself is not 3-connected (neither is its Freudenthal compactification).

Thomassen and Vella [22] proved that:

1. if H is a closed subspace of a compact graph-like space, then H is a graph-like space;
2. a compact graph-like space is locally connected; and
3. graph-like spaces satisfy Menger's Theorem: if k is a non-negative integer and u, v are vertices of G, then either there are $k+1$ internally disjoint uv-arcs in G, or there is a set S of k points, different from u and v, so that $G-S$ has no uv-arc.

The relevance of the first two (each given a relatively short proof in [22]) is that every closed connected subspace of a compact graph-like space is arcwise connected. For the Freudenthal compactification, or more generally for the space $\widetilde{G}$, Diestel and Kühn gave a difficult proof [8].

Many cycle-based theorems about finite graphs extend to compact graph-like spaces. This is, however, not the most interesting possibility. Mirroring Thomas' work in infinite graphs [17], many aspects of the Graph Minors Project of Robertson and Seymour "lift" to graph-like spaces. We will have more to say on this point later in this work.

Furthermore, there is small, but increasing, evidence, that results can be proved for quite general topological spaces. Three examples, all related to planarity of compact, locally connected metric spaces, are [13,15,20]. In the discussion on embedding graph-like spaces into surfaces, we will point out one example of a theorem that we have proved for graph-like spaces that might hold for compact, locally connected metric spaces, but has not yet been proven in that more general context.

2. From finite graphs to compact graph-like spaces

In this section, we review some basic matters concerning cycle spaces and embeddings in surfaces of compact graph-like spaces. These results show that in many basic respects, graph-like spaces have the same properties as graphs.

We denote by 2^E the set of all subsets of edges of the compact graph-like space G. A subset A of 2^E is *thin* if every edge is in only finitely many members of A. In the case A is thin, the *thin sum of* A is the set $\sum_{a\in A} a$ of edges that are in an odd number of elements of A. For a subset A of 2^E, the *subspace generated by* A is the smallest subset of 2^E containing A and closed under thin sums. It is an easy consequence of Zorn's Lemma that every subset of 2^E generates a unique subspace.