Discrete Mathematics 311 (2011) 1472–1496

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Extremal infinite graph theory

Maya Stein

Centro de Modelamiento Matemático, Universidad de Chile, Blanco Encalada, 2120, Santiago, Chile

ARTICLE INFO

Article history:
Available online 22 January 2011

Keywords:
Extremal graph theory
Infinite graph
Connectivity
Degree
Minimum degree
Minor
Minimal k-connectivity

ABSTRACT

We survey various aspects of infinite extremal graph theory and prove several new results. The lead role play the parameters connectivity and degree. This includes the *end degree*. Many open problems are suggested.

1. Introduction

1.1. A short overview

Until now, extremal graph theory usually meant *finite* extremal graph theory. New notions, as the *end degrees* [6,41], *circles* and *arcs*, and the topological viewpoint [10], make it possible to create the infinite counterpart of the theory. We attempt here to give an overview of results and open problems that fall into this emerging area of infinite graph theory.

The paper divides into three parts. The first part is about forcing substructures with assumptions on the degree. We use the vertex-/edge-degree of the ends (for a definition see below) to force highly connected subgraphs and grid minors. For ensuring large complete minors, the vertex-/edge-degree is not enough, and we introduce a new notion, the relative degree, which accomplishes the task, at least for locally finite graphs. Related problems will be addressed along the way.

The second part is on minimal higher connectivity and edge-connectivity of graphs, that is k-(edge-)connectivity for some $k \in \mathbb{N}$. This includes minimality with respect to edge deletion, with respect to vertex deletion, and with respect to taking subgraphs. The main questions here are the existence of vertices or ends of small degree, and bounds on the number of these. We also discuss whether minimal k-connected subgraphs in the sense(s) above exist in every k-connected graph. This will lead to a discussion of the problem in certain subspaces of the topological space associated to the graph.

The third and last part of the survey is on circles and arcs. These are the topologically defined analogues of cycles and paths in infinite graphs (see Section 2). We first discuss results and problems related to Hamilton circles, then move on to (topological) tree-packing and arboricity, and finally close the paper with a discussion of problems related to connectivity-preserving arcs and circles.

1.2. Structure and degree

Extremal graph theory in its strictest sense is all about forcing some palpable properties of a graph, very often some interesting substructure, by making assumptions on the overall density of the graph, conveniently expressed in terms of global parameters such as the average or minimum degree. Turán's well-known theorem is a classical result in this direction, for a discussion of its extension to infinite graphs; see [3,42].

E-mail address: mstein@dim.uchile.cl.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.12.018

Another typical and important result in finite extremal graph theory, which can be found in any standard textbook, is the following theorem of Kostochka. It says that large average degree forces a large complete minor (and the function $f_1(k)$ is essentially the best possible bound [46]).

Theorem 1.2.1 (*[10]*)**.** *There is a constant c_1 so that, for every $k \in \mathbb{N}$, if G is a finite graph of average degree at least $f_1(k) := c_1 k\sqrt{\log k}$, then G has a complete minor of order k.*

Also large topological minors can be forced with similar assumptions in finite graphs, as the following result, due to Bollobás and Thomason, states.

Theorem 1.2.2 (*[10]*)**.** *There is a constant c_2 so that, for every $k \in \mathbb{N}$, if G is a finite graph of average degree at least $f_2(k) := c_2 k^2$, then G has a complete topological minor of order k.*

Let us see how these results could extend to infinite graphs. First of all we have to note that it is not clear what the average degree of an infinite graph should be. We shall thus stick to the minimal degree as our 'density-indicating' parameter. A minor, on the other hand, is defined in same way as for finite graphs, only that the branch-sets may now be infinite.[1]

In rayless graphs we will then get a verbatim extension of Theorems 1.2.1 and 1.2.2 (namely Theorem 3.1.1). This theorem will follow from a useful reduction theorem (Theorem 3.1.2), which states that every rayless graph of minimum degree k has a finite subgraph of minimum degree k. These results will be presented in Section 3.1.

In graphs with rays, however, large minimal degree at the vertices is too weak to force any interesting substructure. This is so because infinite trees may have arbitrarily large degrees, but they do not even have any 2-connected subgraphs. So at first sight, our goal seems unreachable. At second thought, however, the example of the infinite tree just shows that we did not translate the term 'large local densities' in the right way to infinite graphs. Only having every finite part of an infinite graph send out a large number of edges will not produce large overall density, if we do not require something to 'come back' from infinity.

The most natural way to do this is to impose a condition on the ends of the graph. Ends are defined as the equivalence classes of rays (one-way infinite paths), under the equivalence relation of not being separable by any finite set of vertices. Ends have a long history, see [24].

In [6] and in [41], *end degrees* were introduced. In fact, two notions have turned out useful (for different purposes): the vertex-degree and the edge-degree of an end ω. The *vertex-degree* of ω is defined as the maximum cardinality of a set of (vertex-)disjoint rays in ω, and the *edge-degree* is defined as the maximum cardinality of a set of edge-disjoint rays in ω. These maxima exist [19].

Do these notions help to force density in infinite graphs? To some extent they do: A large minimum degree at the vertices together with a large minimum vertex-/edge-degree at the ends implies a certain dense substructure, which takes the form of a highly connected or edge-connected subgraph.

More precisely, there is a function f_v such that every graph of minimum degree resp. vertex-degree $f_v(k)$ at the vertices and the ends has a k-connected subgraph, and there is also a function f_e such that every graph of minimum degree/edge-degree $f_e(k)$ at the vertices and the ends has a k-edge-connected subgraph. While f_e is linear, f_v is quadratic, and this is almost best possible. All these results are from [41] and will be presented in Section 3.3.

Related results will be discussed in Sections 3.2 and 3.4. In the latter, we shall see that in locally finite vertex-transitive graphs, k-connectivity is implied by much weaker assumptions. In fact, the k-(edge-)connectivity of a locally finite vertex-transitive graph is equivalent to all its ends having vertex-(resp. edge-) degree k. In Section 3.2 we shall see that independently of the degrees at the vertices, large vertex-degrees at the ends force an interesting planar substructure: An end of infinite vertex-degree produces the $\mathbb{N} \times \mathbb{N}$-grid as a minor (this is an old result of Halin [19]), and an end of vertex-degree at least $\frac{3}{2}k - 1$ forces a $[k] \times \mathbb{N}$-grid-minor (and this bound is best possible). The latter result was not known before.

However, our notion of vertex-/edge-degrees is not strong enough to make extensions of Theorems 1.2.1 and 1.2.2 possible. This can be seen by taking the infinite r-regular tree and inserting the edge set of some spanning subgraph at each level (Example 3.5.1). With a little more effort we can transform our example into one with infinitely many ends of large but finite vertex-/edge-degree (Example 3.5.2).

To overcome this problem, we introduce in Section 3.6 a new end degree notion, the *relative degree*, that allows us to extend Theorems 1.2.1 and 1.2.2 to infinite locally finite graphs (Theorem 3.6.2). Moreover, every locally finite graph of minimum degree/relative degree at least k has a finite subgraph of average degree at least k (Theorem 3.6.1). An application of Theorem 3.6.2 is investigated in Section 3.7, where we ask whether as in finite graphs, large girth can be used for forcing large complete minors.

1.3. Minimal k-connectivity

The subjects of the second part of our survey are minimally k-connected graphs. Minimality may here mean minimality with respect to either edge or vertex deletion, and it may also mean minimality with respect to taking subgraphs. Minimality

[1] As long as our minors are locally finite, however (which will always be the case in this paper), it does not make any difference whether we allow infinite branch-sets or not. It is easy to see that any infinite branch-set of a locally finite minor may be restricted to a finite one.