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Large sets of resolvable idempotent Latin squares[☆]

Junling Zhou[a], Yanxun Chang[a,*], Zihong Tian[b]

[a] Institute of Mathematics, Beijing Jiaotong University, Beijing 100044, China

[b] Institute of Mathematics, Hebei Normal University, Shijiazhuang 050016, China

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ABSTRACT

An idempotent Latin square of order v is called resolvable and denoted by RILS(v) if the $v(v-1)$ off-diagonal cells can be resolved into $v-1$ disjoint transversals. A large set of resolvable idempotent Latin squares of order v, briefly LRILS(v), is a collection of $v-2$ RILS(v)s pairwise agreeing on only the main diagonal. In this paper we display some recursive and direct constructions for LRILSs.

1. Preliminaries

A *Latin square of order* v LS(v) is a $v \times v$ array in which each cell contains a single symbol from a v-set X, such that each symbol occurs exactly once in each row and exactly once in each column. We usually index the rows, columns, and symbols of an LS(v) by the same v-set. An *incomplete Latin square* LS($v; h_1, h_2, \ldots, h_k$) is a $v \times v$ array L with entries from a v-set X, together with $H_i \subset X$ for $1 \le i \le k$ where $|H_i| = h_i$ and $H_i \cap H_j = \emptyset$ for any $1 \le i < j \le k$. Moreover, (1) each cell of L is empty or contains an element of X; (2) for each $1 \le i \le k$, the subarray indexed by $H_i \times H_i$ is empty (each H_i is called a *hole*); and (3) the elements in row or column x are exactly those of $X \setminus H_i$ if $x \in H_i$, and of X otherwise. If each $|H_i| = h$, $1 \le i \le k$, we denote the incomplete Latin square by LS($v; h^k$). An (incomplete) Latin square $L = (a_{ij})$ is *idempotent* if $a_{ii} = i$ for each i not in the holes. An ILS(v) denotes an idempotent LS(v).

Let L be an (incomplete) Latin square of order v on a symbol set X. A *transversal* of L is a set of v cells, one from each row and column, containing each of the v symbols exactly once. If L is an ILS(v), then the main diagonal cells form a transversal, which we call an *idempotent transversal*. If the $v(v-1)$ off-diagonal cells of an ILS(v) can be resolved into $v-1$ disjoint transversals $T_1, T_2, \ldots, T_{v-1}$, then the ILS($v$) is called *resolvable* and $\Gamma = \{T_0, T_1, \ldots, T_{v-1}\}$ is called a *resolution*, where T_0 is the idempotent transversal. A resolvable ILS(v) is denoted by RILS(v).

Suppose that $L = (a_{ij})$ and $L' = (b_{ij})$ are LS(v)s on a set X. L and L' are *orthogonal* if every element of $X \times X$ occurs exactly once among the v^2 pairs (a_{ij}, b_{ij}), $i, j \in X$. If an idempotent Latin square $L = (a_{ij})$ on I_v is resolvable with v disjoint transversals $T_0, T_1, \ldots, T_{v-1}$, then it has an orthogonal mate $L' = (b_{ij})$, where $b_{ij} = k$ if $(i, j) \in T_k$. If T_0 is the idempotent transversal of L, then the main diagonal entries of L' are all 0's. For an RILS we often use such an orthogonal mate to designate its resolution.

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* Corresponding author.
E-mail addresses: jlzhou@bjtu.edu.cn (J. Zhou), yxchang@bjtu.edu.cn (Y. Chang), tianzh68@163.com (Z. Tian).

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Let $L = (a_{ij})$ and $L' = (b_{ij})$ be two idempotent (incomplete) Latin squares on a set X. They are said to be *disjoint* if $a_{ij} \neq b_{ij}$ for any $i, j \in X$, $i \neq j$ (and i, j not in the same hole). A set of $v - 2$ pairwise disjoint idempotent Latin squares on a v-set X is called a *large set* and denoted by LILS(v). By Chang [3,4,12], there exists an LILS(v) for every $v \geq 3$ and $v \neq 6$. A large set of resolvable ILS(v)s, denoted by LRILS(v), is an LILS(v) with each member ILS(v) resolvable.

Let X be a v-set. A *t-wise balanced design* (t-BD) of order v is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a family of subsets of X (called *blocks*) such that each t-subset of X is contained in exactly one block of $\mathcal{A}$. An $S(t, K, v)$ denotes a t-BD of order v with block sizes from the set K. A *group divisible t-design* (or t-GDD) on a v-set X is a triple $(X, \mathcal{G}, \mathcal{B})$ satisfying the following properties: (1) $(X, \mathcal{G})$ is a 1-BD (the elements of $\mathcal{G}$ called *groups*); (2) $\mathcal{B}$ is a family of subsets of X (called blocks) and each block intersects any given group in at most one point; (3) each t-subset from t distinct groups is contained in exactly one block. If the block sizes come from a set K, we denote the GDD by GDD(t, K, v). The *type* of the GDD means the list $\{|G| : G \in \mathcal{G}\}$. If the blocks of a GDD$(t, K, v)$ on X can be partitioned into some *parallel classes*, each containing every element of X exactly once, then the GDD is called *resolvable* and abbreviated to RGDD. Whenever $K = \{k\}$, we often write an $S(t, K, v)$ as $S(t, k, v)$ and a GDD(t, K, v) as GDD(t, k, v). A GDD$(2, k, kt)$ of type t^k is called a *transversal design* and denoted by TD(k, t). A resolvable TD(k, t) is denoted by RTD(k, t).

An alternative representation of an idempotent Latin square can be useful. An *ordered design* OD(v) on a v-set X is a $3 \times v(v - 1)$ array such that (1) each column has 3 distinct elements of X; and (2) each two rows contains each ordered pair of distinct elements of X precisely once. An OD(v) on X is often displayed as a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is the collection of the column vectors (often expressed in their transposes). Let $L = (l_{ij})$ be an ILS(v) defined on X. Taking (i, j, l_{ij}), $i, j \in X$ and $i \neq j$, as the columns, then we represent the ILS(v) as an OD(v). Conversely, by supplementing v idempotent columns, an OD(v) corresponds to an ILS(v). Further if an ILS(v) is resolvable with a resolution $\{T_0, T_1, \ldots, T_{v-1}\}$ where T_0 is the idempotent transversal, then the corresponding OD(v) is also called resolvable (and denoted by ROD(v)) with $v - 1$ transversals $S_k = \{(i, j, l_{ij}) : (i, j) \in T_k\}$, $1 \leq k \leq v - 1$. Likewise, an LRILS(v) corresponds to an LROD(v), a large set of $v - 2$ pairwise disjoint resolvable OD(v)s. Similarly, an idempotent incomplete Latin square ILS$(v; h_1, h_2, \ldots, h_k)$ can be regarded as a partial ordered design and indicated by $(X, \mathcal{G}, \mathcal{A})$ where X is the v-set, $\mathcal{G}$ is the set of the k holes, and $\mathcal{A}$ is the collection of the column vectors.

We typically blur the distinctions between ILS(v) and OD(v), and choose a convenient expression in the subsequent sections. The main purpose of this paper is to study the existence problem of LRILS, or equally LROD. Obviously there does not exist any LRILS(6) since no LILS(6) exists. It is worth to mention that some research on large sets of resolvable triple systems, including large sets of Kirkman triple systems (LKTSs) and large sets of resolvable Mendelsohn triple systems (LRMTSs), has been done. We would not dwell on the detailed explanation of LKTS and LRMTS, but rather mention an obvious fact that the existence of an LKTS(v) or an LRMTS(v) implies the existence of an LRILS(v). So we can transfer the known results on LKTSs and LRMTSs to LRILSs.

Theorem 1.1. *There exists an LRILS(v) for any of the following orders:*

(1) $v = 7^k + 2, 13^k + 2, 25^k + 2, 2^{4k} + 2$ *and* $2^{6k} + 2$ *where* $k \geq 0$;
(2) $v = 3r$, *where* $r \in \{1, 4, 5, 7, 8, 11, 12, 13, 16, 17, 20, 23, 25, 28, 32, 35, 36, 37, 40, 41, 43, 47, 53, 55, 57, 60, 61, 65, 67, 68, 71, 76, 77, 84, 91, 92, 93, 95, 97, 100, 103, 113, 121, 123\} \bigcup \{2^{2p+1}25^q + 1 : p, q \geq 0\} \bigcup \{\prod_{i=1}^{s}(2q_i^{n_i} + 1) \prod_{j=1}^{t}(4^{m_j} - 1) : s + t \geq 1, n_i, m_j \geq 1\ (1 \leq i \leq s, 1 \leq j \leq t), q_i \equiv 7 \pmod{12}$ *and* q_i *is a prime power*$\}$.

Furthermore, for any order v in (1) and (2) with $v \not\equiv 2 \pmod 4$, there exists an LRILS $(v 3^a 5^b \Pi_{i=1}^{s}(2 \cdot 13^{n_i} + 1) \Pi_{j=1}^{t}(2 \cdot 7^{m_j} + 1))$, *where* $n_i, m_j \geq 1 (1 \leq i \leq s, 1 \leq j \leq t)$, $a, b, s, t \geq 0$ *and* $a + s + t \geq 1$.

Proof. The parallel results to (1) and (2) for large sets of Kirkman triple systems or resolvable Mendelsohn triple systems can be found in [7,13]. Then we apply the product constructions in [15,16] to get more infinite classes, where we utilize the existence of LR-designs in [6]. □

An LRILS(q) is easy to prove to exist if $q \geq 3$ is a prime power.

Lemma 1.2. *There exists an LRILS* (q) *for any prime power* $q \geq 3$.

Proof. By [1], for any prime power $q \geq 3$ there exist $q - 1$ mutually orthogonal Latin squares of order q on I_q. By some permutations of rows and columns, we can form $q - 1$ new mutually orthogonal Latin squares, say $L_1, L_2, \ldots, L_{q-1}$, in such a way that the main diagonal entries of L_{q-1} are all 0's. Accordingly, the main diagonal of each L_i $(1 \leq i \leq q-2)$ is a transversal. By renaming the symbols of L_i, we obtain $q - 2$ mutually orthogonal idempotent Latin squares L_i' $(1 \leq i \leq q - 2)$. They are resolvable and pairwise agree only on the main diagonal and hence form an LRILS(q). □

By Lemma 1.2, the smallest LRILS(v) in doubt is for $v = 10$. We illustrate an LRILS(10) in the following lemma.

Lemma 1.3. *There exists an LRILS* (10).

Proof. Let α be a primitive element of $GF(8)$ such that $\alpha^3 + \alpha + 1 = 0$ and $GF(8) = \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$. In short we denote the elements $\alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha$, and $\alpha^2+\alpha+1$ by 2, 3, 4, 5, 6, and 7 respectively. An RILS(10) L_0 is constructed as follows on $X = GF(8) \bigcup \{a, b\}$, with the rows and columns indexed in the order of $0, 1, \ldots, 7, a, b$. Its