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List injective colorings of planar graphs ☆

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ABSTRACT

A vertex coloring of a graph G is called injective if any two vertices joined by a path of length two get different colors. A graph G is injectively k-choosable if any list L of admissible colors on $V(G)$ of size k allows an injective coloring φ such that $\varphi(v) \in L(v)$ whenever $v \in V(G)$. The least k for which G is injectively k-choosable is denoted by $\chi_i^l(G)$.

Note that $\chi_i^l \geq \Delta$ for every graph with maximum degree Δ. For planar graphs with girth g, Bu et al. (2009) [15] proved that $\chi_i^l = \Delta$ if $\Delta \geq 71$ and $g \geq 7$, which we strengthen here to $\Delta \geq 16$. On the other hand, there exist planar graphs with $g = 6$ and $\chi_i^l = \Delta + 1$ for any $\Delta \geq 2$. Cranston et al. (submitted for publication) [16] proved that $\chi_i^l \leq \Delta + 1$ if $g \geq 9$ and $\Delta \geq 4$. We prove that each planar graph with $g \geq 6$ and $\Delta \geq 24$ has $\chi_i^l \leq \Delta + 1$.

1. Introduction

By a graph we mean a non-oriented graph without loops and multiple edges. By $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively, and let $g(G)$ and $\Delta(G)$ be the girth and maximum degree of G, respectively. (We shall drop the argument whenever the graph is clear from context.)

A vertex coloring of a graph G is called *injective* if any two vertices having a common neighbor get different colors. Let $\chi_i(G)$ be the minimum number of colors in injective colorings of G. Clearly, $\chi_i(G) \geq \Delta(G)$ for each G, where $\Delta(G)$ is the maximum degree of G. The injective coloring is originated in complexity theory, and is used in coding theory [18]. Note that the injective coloring is not necessarily proper, and this is its only difference from the 2-distance coloring.

The 2-distance coloring is a special case $p = q = 1$ of the (p, q)-coloring, which is one of the most natural models in the frequency assignment problem in mobile phoning: the vertices of a planar graph (sources) should be colored (get frequencies) so that the colors (integers) of vertices at distance 1 differ by at least p, while those at distance 2, by at least q. In practice, $p \geq q$, for the interference decreases as the distance increases. Sometimes, the set of allowed frequencies can vary from one source to another; this corresponds to the list (p, q)-coloring, i.e., the (p, q)-choosability. In particular, [3,14] give upper and lower bounds for the list (p, q)-chromatic number of planar graphs with large enough girth that differ by an additive constant not depending on the main parameter, p.

Let $\chi_2(G)$ be the minimum number of colors in 2-distance colorings of G. In 1977, Wegner [24] conjectured that $\chi_2 \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ for every planar graph with $\Delta \geq 8$. The following upper bounds have been established: $\lfloor \frac{9\Delta}{5} \rfloor + 2$ for $\Delta \geq 749$ by Agnarsson and Halldórsson [1,2] and $\lceil \frac{9\Delta}{5} \rceil + 1$ for $\Delta \geq 47$ by Borodin et al. [4,5]. The best published upper bounds for large Δ are due to Molloy and Salavatipour [22,23]: $\lceil \frac{5\Delta}{3} \rceil + 78$ for all Δ and $\lceil \frac{5\Delta}{3} \rceil + 25$ for $\Delta \geq 241$.

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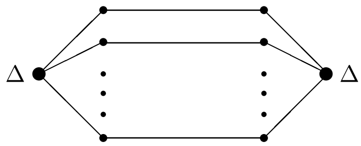

Fig. 1. $g = 6$ and $\chi_i = \Delta + 1$ for $\Delta \geq 2$.

Clearly, $\chi_2(G) \geq \Delta(G) + 1$ for every graph G. In [10,6,20], we give sufficient conditions (in terms of g and Δ) for the 2-distance chromatic number of a planar graph to equal the trivial lower bound $\Delta + 1$. In particular, we determine the least g such that $\chi_2 = \Delta + 1$ if Δ is large enough (depending on g) to be equal to 7. Putting it differently, there exist planar graphs with $g \leq 6$ such that $\chi_2 = \Delta + 2$ for arbitrarily large Δ (this was proved independently by Dvořák et al. in [17]). On the other hand, Borodin et al. [12,13] proved that $\chi_2 = \Delta + 1$ whenever $\Delta \geq 31$ for planar graphs of girth six under the additional assumption that each edge is incident with a vertex of degree two. Dvořák et al. [17] proved that every planar graph with $\Delta \geq 8821$ and $g \geq 6$ has $\chi_2 \leq \Delta + 2$; in [7–9], we lowered the restriction on Δ here to 18, and for the 2-distance choosability, to 24.

As compared to the 2-distance coloring, the injective coloring of planar graphs has been studied much less. Note that for triangle-free graphs, this coloring is just the case $p = 0, q = 1$ of the (p, q)-coloring. Also observe that the ordinary (proper) vertex coloring can be regarded as the $(1, 0)$-coloring, and so the injective coloring is opposite to the ordinary coloring from the viewpoint of the (p, q)-coloring.

It is easy to construct a planar graph with $g = 4$ and arbitrary even Δ with $\chi_i = \frac{3}{2}\Delta$. Clearly, each graph G has $\chi_i(G) \geq \Delta(G)$, so it seems natural to try to describe graphs having $\chi_i = \Delta$. For planar graphs, the following sufficient conditions are known: $\Delta \geq 71$ and $g \geq 7$ [15], $\Delta \geq 4$ and $g \geq 13$ [16], and $\Delta \geq 3$ and $g \geq 19$ [21].

If every vertex v of G has its own set $L(v)$ of admissible colors, where $|L(v)| = k$, then we say that $V(G)$ has a *list L of size k*. A graph G is said to be *injectively k-choosable* if any list L of size k allows an injective coloring φ such that $\varphi(v) \in L(v)$ whenever $v \in V(G)$. The least k for which G is injectively k-choosable is *the injective choosability number of* G, denoted by $\chi_i^l(G)$. Note that $\chi_i^l(G) \geq \chi_i(G)$ for every graph G.

The first purpose of this paper is to show that each planar graph G with $g(G) \geq 7$ and large enough $\Delta(G)$ has $\chi_i^l(G) = \Delta(G)$:

Theorem 1. *If G is a planar graph of girth g and maximum degree Δ, then* $\chi_i^l(G) = \chi_i(G) = \Delta$ *in each of the cases* (i–iv)*:*

(i) $\Delta \geq 16$ *and* $g = 7$;
(ii) $\Delta \geq 10$ *and* $8 \leq g \leq 9$;
(iii) $\Delta \geq 6$ *and* $10 \leq g \leq 11$;
(iv) $\Delta = 5$ *and* $g \geq 12$.

On the other hand, a cycle C_{4n-1} has $\chi_i(C_{4n-1}) = \Delta(C_{4n-1}) + 1 = 3$, while the graph in Fig. 1 has $g = 6$ and $\chi_i(G) = \Delta + 1$ for arbitrarily large Δ.

For planar graphs, the following results are also known: $\chi_i \leq \Delta + 4$ if $g = 5$ [21], $\chi_i \leq \Delta + 2$ if $g \geq 8$ [15], $\chi_i \leq \Delta + 1$ if $g \geq 9$ and $\Delta \geq 4$ [15,16,21].

The second purpose of this paper is to prove a sharp upper bound for the injective choosability number of planar graphs with girth six:

Theorem 2. *Every planar graph G with* $\Delta(G) \geq 24$ *and* $g(G) \geq 6$ *has* $\chi_i^l(G) \leq \Delta(G) + 1$.

We would like to attract attention to the following

Problem 1. Find a precise upper bound for the injective choosability number of planar graphs with given girth and maximum degree.

Due to Theorems 1 and 2, for $g \geq 6$ this problem is open only if Δ is relatively small (depending on g). As far as we know, Problem 1 remains completely open for $g \leq 5$.

Our proof of Theorem 1 is based on the idea of 2-alternating cycle introduced in [3] and used, in particular, in [6,11,14,19] for solving various problems on the coloring and edge-partition of planar graphs. In the proofs of both Theorems 1 and 2 (given in Sections 3 and 4, respectively), we also employ the techniques of face-transmitters elaborated in [7,9,8,20].

2. Common preliminaries to proving Theorems 1 and 2

Assuming the contrary, we first find a counterexample G to Theorems 1 or 2 with the minimum number of edges, as follows.

Let G' be any counterexample, say, to Theorem 1, i.e., we have $\Delta(G') = \Delta \geq 5$, $g(G')$ is at least as large as in the corresponding item of Theorem 1, while $\chi_i^l(G') \geq \Delta + 1$. Now let G be a graph with the fewest edges such that $\Delta(G) \leq \Delta$,