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Dominating direct products of graphs

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Abstract

An upper bound for the domination number of the direct product of graphs is proved. It in particular implies that for any graphs G and H, $\gamma(G \times H) \leqslant 3\gamma(G)\gamma(H)$. Graphs with arbitrarily large domination numbers are constructed for which this bound is attained. Concerning the upper domination number we prove that $\Gamma(G \times H) \geqslant \Gamma(G)\Gamma(H)$, thus confirming a conjecture from [R. Nowakowski, D.F. Rall, Associative graph products and their independence, domination and coloring numbers, Discuss. Math. Graph Theory 16 (1996) 53–79]. Finally, for paired-domination of direct products we prove that $\gamma_{\mathrm{pr}}(G \times H) \leqslant \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H)$ for arbitrary graphs G and H, and also present some infinite families of graphs that attain this bound.

MSC: 05C69; 05C12

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1. Introduction

Many authors have investigated the behavior of domination and independence parameters in graph products, cf. the survey by Nowakowski and Rall [15]. In this paper we focus on domination in direct products of graphs that was initiated in [6,15].

There is no consistent relation between the domination number of the direct product of two graphs and the product of their domination numbers [15]. In the same paper an example was given that disproved a Vizing-type conjecture from [6]. An infinite series of such examples was presented in [11]. Chérifi et al. [3] determined the domination number of the direct product of two paths with the only exception when one factor is a path on 10, 11, or 13 vertices. Independently, some of these results were obtained by Klobučar in [12,13]. In [15] it was proved that $\gamma(G \times H) \geqslant \rho(G)\gamma_t(H)$, see also [17] for a shorter proof, while in [17] it was observed that

$$\gamma(G \times H) \leqslant 4\gamma(G)\gamma(H).$$

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Other domination-type problems in direct products were studied in [4,5,9,14,19]. In this note we further investigate the domination number, and present, to our knowledge, the first results on the paired-domination and upper domination numbers of direct products of graphs.

In the next section we define the concepts needed and recall two known results to be used later. In Section 3 we prove an upper bound on $\gamma(G \times H)$ that in particular implies that

$$\gamma(G \times H) \leqslant 3\gamma(G)\gamma(H).$$

We also construct graphs G and H with arbitrarily large domination numbers for which this bound is attained and show that in many cases it can be further improved. In the fourth section we prove that for any graphs G and H,

$$\Gamma(G \times H) \geqslant \Gamma(G)\Gamma(H),$$

where $\Gamma(G)$, as usual, denotes the upper domination number of a graph G. This result was conjectured in [15]. In the last section we consider paired-domination in direct products. We show that for any graphs G and H

$$\gamma_{\mathrm{pr}}(G \times H) \leqslant \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H),$$

present some infinite families of graphs for which the equality is attained and pose a problem concerning a possible lower bound for $\gamma_{\mathrm{pr}}(G \times H)$.

2. Preliminaries

All the graphs considered will be simple, undirected graphs with no isolated vertices. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S. If, in addition, each vertex of S has a neighbor in S, then S is called a *total dominating set*. Furthermore, as in [8], a dominating set S is called a *paired-dominating set* if it induces a subgraph with a perfect matching. Equivalently, S is the set of endvertices of a matching of G that is also a dominating set of G. If A and B are subsets of V we say that A *dominates* B if every vertex of B has a neighbor in A or is a vertex of A. We also say that B is *dominated* by A. We will make use of the following well-known result.

Theorem 1 (*Ore [16]*). *A dominating set S of a graph G is minimal if and only if for every vertex $u \in S$ one of the following two conditions holds*:

(i) *u is not adjacent to any vertex of S*,
(ii) *there exists a vertex $v \in V(G) \setminus S$ such that u is the only neighbor of v from S.*

The *domination* (resp. *total domination, paired-domination*) *number* $\gamma(G)$ (resp. $\gamma_{\mathrm{t}}(G)$, $\gamma_{\mathrm{pr}}(G)$) of a graph G is the minimum cardinality of a dominating (resp. total dominating, paired-dominating) set. A dominating set of size $\gamma(G)$ is called a γ-*set*. Analogously we define a γ_{t}-*set* and a γ_{pr}-*set*. Note that for any graph G we have $\gamma(G) \leqslant \gamma_{\mathrm{t}}(G) \leqslant \gamma_{\mathrm{pr}}(G)$. The *upper domination number* $\Gamma(G)$ of G is the size of a largest minimal dominating set. The 2-*packing number* $\rho(G)$ of G is the maximum cardinality of a set $S \subset V(G)$ such that any two vertices in S are at distance at least three. Equivalently, the vertices of S have pairwise disjoint closed neighborhoods. The *open packing number* $\rho^o(G)$ is the maximum cardinality of a set of vertices whose open neighborhoods are pairwise disjoint. For more information on domination in graphs we refer to [7].

For graphs G and H, the *direct product* $G \times H$ (also known as the tensor product, cross product, cardinal product, categorical product, ...) is the graph with vertex set $V(G) \times V(H)$ where two vertices (x, y) and (v, w) are adjacent if and only if $xv \in E(G)$ and $yw \in E(H)$. The *Cartesian product* $G \square H$ is defined on the same vertex set where two vertices (x, y) and (v, w) are adjacent if and only if either $x = v$ and $yw \in E(H)$, or $y = w$ and $xv \in E(G)$.

We will also need the following result from [1].

Lemma 2. *Let $X = G \times H$ and let (x, y), (v, w) be vertices of X. Then $d_X((x, y), (v, w))$ is the smallest d such that there is an x, v-walk of length d in G and a y, w-walk of length d in H.*