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Blockers and transversals in some subclasses of bipartite graphs: When caterpillars are dancing on a grid

B. Ries [b], C. Bentz [e], C. Picouleau [c], D. de Werra [d], M.-C. Costa [f], R. Zenklusen [a]

[a] ETHZ, Eidgenössische Technische Hochschule Zürich, Zürich, Switzerland
[b] Columbia University, New York, USA
[c] CNAM, Laboratoire CEDRIC, Paris, France
[d] EPFL, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
[e] Université Paris-Sud, LRI, Orsay, France
[f] ENSTA, UMA-CEDRIC, Paris, France

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ABSTRACT

Given an undirected graph $G = (V, E)$ with matching number $\nu(G)$, a d-blocker is a subset of edges B such that $\nu((V, E \setminus B)) \leq \nu(G) - d$ and a d-transversal T is a subset of edges such that every maximum matching M has $|M \cap T| \geq d$. While the associated decision problem is NP-complete in bipartite graphs we show how to construct efficiently minimum d-transversals and minimum d-blockers in the special cases where G is a grid graph or a tree.

1. Introduction

Given a collection $\mathcal{C}$ of subsets of a ground set E we define a transversal as a subset of E which meets every member of $\mathcal{C}$. Transversals have an interest for themselves but also for their numerous applications (for instance surveying when $\mathcal{C}$ is a collection of paths from the origin to destination in a graph). Such sets have been extensively studied for various collections $\mathcal{C}$ (see for instance [3] and the chapter 22 in [7]).

A family of problems which follows a similar spirit to transversal problems are the class of edge deletion problems [1,4,6,8]. In [10] a generalization of transversals called (d-transversals) and a closely related edge deletion problem (d-blockers) were introduced for matchings; complexity results have been derived and some polynomially solvable cases have been presented.

For general bipartite graphs finding minimum d-blockers or d-transversals is NP-hard [10]. Our purpose in this paper is to show how minimum d-transversals and minimum d-blockers can be constructed in some specific subclasses of bipartite graphs: the grid graphs and the trees. For trees, the algorithms to be presented will essentially be based on dynamic programming. For grid graphs the technique will be different: the structural simplicity of such graphs will allow us to construct directly d-blockers and d-transversals. Most of the effort will then be spent to show that no smaller subset of

E-mail addresses: br2275@columbia.edu (B. Ries), bentz@lri.fr (C. Bentz), chp@cnam.fr (C. Picouleau), dominique.dewerra@epfl.ch (D. de Werra), marie-christine.costa@ensta.fr (M.-C. Costa), rico.zenklusen@ifor.math.ethz.ch (R. Zenklusen).

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edges can be a d-blocker or d-transversal. Depending on the parity of parameters m and n describing the size of the grid graphs, various proof techniques (bounding procedures) will be necessary.

In Section 2, we will recall the basic definitions of d-blocker and d-transversal as well as some results of [10]; then we will introduce specific notations for grid graphs. In Section 3 we will state and prove the formulas giving the sizes of minimum d-transversals in grid graphs. Section 4 will study minimum d-blockers in grid graphs. Section 5 will be dedicated to the case of trees (minimum d-transversals and minimum d-blockers) and conclusions will follow in Section 6.

2. Definitions and previous results

All graph theoretical terms not defined here can be found in [2]. Throughout this paper we are concerned with undirected simple loopless graphs $G = (V, E)$. A **matching** M is a set of pairwise non-adjacent edges. A matching M is called **maximum** if its cardinality $|M|$ is maximum. The largest cardinality of a matching in G, its **matching number**, will be denoted by $\nu(G)$. Let $\boldsymbol{P_0(G)} = \{[v, w] \in E | \forall$ maximum matching M, $[v, w] \notin M\}$ and $\boldsymbol{P_1(G)} = \{[v, w] \in E | \forall$ maximum matching M, $[v, w] \in M\}$. A vertex $v \in V$ is called **saturated by a matching** M if there exists an edge $[v, w] \in M$. A vertex $v \in V$ is called **strongly saturated** if for all maximum matchings M, v is saturated by M. We denote by $\boldsymbol{S(G)}$ the set of strongly saturated vertices of a graph G. We will be interested in subsets of edges which will intersect maximum matchings in G or whose removal will reduce the matching number by a given number.

We shall say that a subset $T \subseteq E$ is a **d-transversal** of G if for every maximum matching M we have $|M \cap T| \geq d$. Thus a d-transversal is a subset of edges which intersect each maximum matching in at least d edges.

A subset $B \subseteq E$ will be called a **d-blocker** of G if $\nu(G') \leq \nu(G) - d$ where G' is the partial graph $G' = (V, E \setminus B)$. So B is a subset of edges whose removal reduces the cardinality of a maximum matching by at least d.

In case where $d = 1$, a d-transversal or a d-blocker is called a **transversal** or a **blocker**, respectively. We remark that in this case our definition of a transversal coincides with the definition of a transversal in the hypergraph of maximum matchings of G.

We denote by $\beta_{\boldsymbol{d}}(\boldsymbol{G})$ the minimum cardinality of a d-blocker in G and by $\tau_{\boldsymbol{d}}(\boldsymbol{G})$ the minimum cardinality of a d-transversal in G ($\beta(G)$ and $\tau(G)$ in case of a blocker or a transversal). A d-blocker (resp. d-transversal) will be *minimum* if it is of minimum size.

Let v be a vertex in graph G. The **bundle** of v, denoted by $\omega(v)$, is the set of edges which are incident to v. So $|\omega(v)| = d(v)$ is the degree of v. As we will see, bundles play an important role in finding d-transversals and d-blockers.

A **grid graph** (or shortly a **grid**) $G_{m,n} = (V, E)$ is constructed on vertices x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$; its edge set consists of horizontal edges $h_{ij} = [x_{ij}, x_{i,j+1}]$, $1 \leq j \leq n - 1$ in each row i, $1 \leq i \leq m$, and of vertical edges $v_{ij} = [x_{ij}, x_{i+1,j}]$, $1 \leq i \leq m - 1$, in each column j, $1 \leq j \leq n$.

Notice that $G_{m,n}$ is a bipartite graph; let $\mathcal{B}$, $\mathcal{W}$ be the associated partition of its vertex set. When mn is even, $|\mathcal{B}| = |\mathcal{W}| = \frac{mn}{2}$ and the maximum matchings are perfect (all vertices are saturated), i.e. $\nu(G_{m,n}) = \frac{mn}{2}$. When mn is odd, assuming that the four corners (vertices of degree 2) are in $\mathcal{B}$, we have $|\mathcal{B}| = \frac{mn+1}{2}$ and $|\mathcal{W}| = \frac{mn-1}{2}$, so $|\mathcal{B}| = |\mathcal{W}| + 1$; every maximum matching will saturate all vertices but one, i.e. $\nu(G_{m,n}) = \lfloor \frac{mn}{2} \rfloor$. Moreover for every vertex v in $\mathcal{B}h$, there is a maximum matching saturating all vertices except v.

We give some properties and results concerning d-transversals and d-blockers (see [10] for their proofs).

Property 2.1. *In any graph G and for any $d \geq 1$, a d-blocker B is a d-transversal.*

Property 2.2. *In any graph $G = (V, E)$ a set T is a transversal if and only if it is a blocker.*

Property 2.3. *For any independent set $\{v_1, v_2, \ldots, v_d\} \subseteq S(G)$ the set $T = \cup_{i=1}^{d} \omega(v_i)$ is a d-transversal.*

For the special case of $G_{1,n}$, a grid graph with a unique row, we have the following.

Property 2.4. *Let $G_{1,n}$ be a chain on n vertices $v_1, v_2, \ldots, v_n$ (i.e., $E = \{[v_i, v_{i+1}] | i = 1, \ldots, n-1\}$) and $d \geq 1$ an integer. Then*

- $\beta_d(G) = 2d - 1$ *and* $\tau_d(G) = d$ *if* n *is even,*
- $\beta_d(G) = \tau_d(G) = 2d$ *if* n *is odd.*

One can observe from the previous property that for the case where n is even and $d > 1$ we have $\tau_d(G) < \beta_d(G)$: so a d-transversal is not necessarily a d-blocker (i.e. the converse of Property 2.1 is not necessarily true).

In case where G is bipartite:

Theorem 2.1. *For every fixed $d \in \{1, 2, \ldots, \nu(G)\}$ finding a minimum d-blocker or a minimum d-transversal is $\mathcal{NP}$-hard even if G is bipartite.*

3. Minimum d-transversal in grid graphs

We show here how to construct a minimum d-transversal in a grid graph $G_{m,n}$. In the case where mn is even, the d-transversals constructed will generally consist of d bundles whose centers form a stable set. In some cases other constructions will be needed.