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Partial duality and Bollobás and Riordan's ribbon graph polynomial

Iain Moffatt

Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA

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ABSTRACT

Recently S. Chmutov introduced a generalization of the dual of a ribbon graph (or equivalently an embedded graph) and proved a relation between Bollobás and Riordan's ribbon graph polynomial of a ribbon graph and of its generalized duals. Here I show that the duality relation satisfied by the ribbon graph polynomial can be understood in terms of knot theory and I give a simple proof of the relation which used the homfly polynomial of a knot.

1. Introduction and motivation

Recently, there has been a lot of interest in connections between knots and ribbon graphs [3–6,15,16]. In particular, there are various constructions which realize the Jones polynomial of a link as an evaluation of Bollobás and Riordan's ribbon graph polynomial (defined in [1,2]) of an associated signed ribbon graph. These constructions generalize Thistlethwaite's seminal result connectng the Tutte and Jones polynomials in [18], and Kauffman's connection between the Jones and dichromatic polynomials in [12]. In [3], Chmutov and Pak proved that the Jones polynomial of a virtual link or a link in a thickened surface is an evaluation of the signed ribbon graph polynomial. In other work in this area, Dasbach et al. in [6] showed how to construct a (non-signed) ribbon graph from a (not necessarily alternating) link diagram with the property that the Jones polynomial is an evaluation of the ribbon graph polynomial of the ribbon graph. Given the similarity between these two results, as they both relate the Jones and ribbon graph polynomials, it is natural to look for a connection between them. This question was first answered in [16] where I defined an "unsigning" procedure which took in a signed plane graph and gave out a non-signed ribbon graph. Chmutov has also considered the relationship between the ribbon graph models for the Jones polynomial, particularly between those in [3,4] (In [4] Chmutov and Voltz extended the results of Dasbach et al. from [6] to virtual links). In the process, Chmutov defined a generalized duality for ribbon graphs (which I call "partial duality" here[1]) of which my unsigning is a special case (as was observed by Chmutov in [5]). Chmutov not only showed that his partial duality connected ribbon graph models for the Jones polynomial, but also that it has desirable properties with respect to the signed ribbon graph polynomial. These desirable properties generalize the well known behavior of the Tutte polynomial under duality. In this paper I am interested in this partial duality and the ribbon graph polynomial.

The partial dual G^A of a ribbon graph G is constructed by forming the dual of a ribbon graph only along the edges in $A \subseteq E(G)$, as described in Section 2.3. Since $G^{E(G)} = G^*$, Poincaré duality is a special case of Chmutov's partial duality. Chmutov proved that, up to a normalization, the signed ribbon graph polynomials of G and G^A are equal when $xyz^2 = 1$.

E-mail address: imoffatt@jaguar1.usouthal.edu.

1 With thanks to Dan Archdeacon for suggesting the name "partial duality".

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In [15], I used the fact that the homfly polynomial determines the ribbon graph polynomial to prove that the ribbon graph polynomials of G and G^* are equal, again up to a normalization, and again along the surface $xyz^2 = 1$. Connecting the facts that Chmutov's duality relation holds along $xyz^2 = 1$; the homfly polynomial determines the ribbon graph polynomial along xyz^2; and a special case ($A = E(G)$) of Chmutov's duality theorem has a simple proof through knot theory, one naturally suspects that Chmutov's duality theorem can be understood in terms of knot theory. Here I show that this is indeed the case, and provide a proof of Chmutov's duality relation using knot theory. Showing that there is a knot theoretical foundation for this result offers a new understanding of the underlying structures of duality and the ribbon graph polynomial.

The argument I use to prove Chmutov's duality theorem is essentially: Step 1: the homfly determines the signed ribbon graph polynomial; Step 2: the links associated with G and G^A have the same homfly polynomial. This two step argument is structured in this paper in the following way. In Section 2 I define the the partial dual of a signed, orientable ribbon graph. In Section 3, I review the definitions of the signed ribbon graph polynomial and the homfly polynomial. I then go on to show how the homfly polynomial determines the signed ribbon graph polynomial along $xyz^2 = 1$. This is split between sections in Section 3.3, where I review results from [15], and Section 4.1, where I express the signed ribbon graph polynomial in terms of the homfly polynomial and reformulate Chmutov's duality theorem. Finally, in Section 4.2, I give a simple proof of the knot theoretic reformulation of the duality theorem.

I would like to thank Tom Zaslavsky for encouraging me to write down these results.

2. The partial dual of a ribbon graph

2.1. Ribbon graphs

Roughly speaking, a ribbon graph is a 'topological graph' formed by using disks as vertices and ribbons $I \times I$ as edges. Ribbon graphs provide a convenient description of cellularly embedded graphs (a cellularly embedded graph is an embedded graph with the property that each of its faces is a 2-cell).

Definition 1. A *ribbon graph* $G = (V(G), E(G))$ is a surface with a boundary represented as the union of closed disks (called *vertices*) and ribbons $I \times I$, where $I = [0, 1]$ is the unit interval, (called *edges*) such that

1. the vertices and edges intersect in disjoint line segments $\{0, 1\} \times I$;
2. each such line segment lies on the boundary of precisely one vertex and precisely one edge;
3. every edge contains exactly two such line segments.

A ribbon graph is said to be *orientable* if its underlying surface is orientable.

A ribbon graph G is said to be *signed* if it is equipped with a mapping from its edge set $E(G)$ to $\{+, -\}$ (so a sign $+$ or $-$ is assigned to each edge of G).

Ribbon graphs are considered up to homeomorphisms of the surface that preserve the vertex-edge structure. Some signed ribbon graphs are shown in Examples 5 and 6.

It is often convenient to label the edges of ribbon graphs. I will often abuse notation and identify an edge with its unique label. At times I will also abuse notation and use e to denote an edge of a ribbon graph and the label of that edge.

It is well known that ribbon graphs are equivalent to cellularly embedded graphs (considered up to homeomorphism of the surface). Details of the equivalence of ribbon graphs and cellularly embedded graphs can be found in [9], for example. Here I will work primarily in the language of ribbon graphs, rather than embedded graphs, as the topology of ribbon graphs is particularly convenient for my purposes.

In this paper I will be primarily interested in orientable ribbon graphs. An orientable ribbon graph is equivalent to a graph cellularly embedded in an orientable surface and is also equivalent to a combinatorial map (that is a graph equipped with a cyclic order of the incident half-edges at each vertex). The restriction here to orientable ribbon graphs is due to that fact that, at the time of writing, the homfly polynomial of a link in a thickened non-orientable surface has yet to be defined. It should be emphasized that all of the graph theoretical constructions used in this paper do work for non-orientable ribbon graphs. Also, I expect that the knot theoretic methods used in this paper would extend to the non-orientable case with a suitable definition of the homfly polynomial of a link in a thickened non-orientable surface.

2.2. Arrow presentations

In order to define partial duality it will convenient to describe ribbon graphs using arrow presentations. Arrow presentations provide a useful combinatorial description of a ribbon graph.

Definition 2. From [5], an *arrow presentation* consists of a set of circles, called *cycles*, equipped with a set of disjoint, labelled arrows marked along their perimeters. Each label appears on precisely two arrows. Two arrow presentations are considered equivalent if one can be obtained from the other by reversing the direction of all of the marking arrows which belong to some subset of labels or by changing the set of labels used.

An arrow presentation is said to be *signed* if there is a mapping from the set of labels of the arrows to $\{+.-\}$.