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A note on traversing specified vertices in graphs embedded with large representativity

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ABSTRACT

A graph G is said to have property $P(2, k)$ if given any $k+2$ distinct vertices $a, b, v_1, \dots, v_k$, there is a path P in G joining a and b and passing through all of $v_1, \dots, v_k$. A graph G is said to have property $C(k)$ if given any k distinct vertices $v_1, \dots, v_k$, there is a cycle C in G containing all of $v_1, \dots, v_k$. It is shown that if a 4-connected graph G is embedded in an orientable surface Σ (other than the sphere) of Euler genus $eg(G, \Sigma)$, with sufficiently large representativity (as a function of both $eg(G, \Sigma)$ and k), then G possesses both properties $P(2, k)$ and $C(k)$.

1. Introduction

If k is a positive integer, graph G is said to have property $C(k)$ if for every set S of k vertices, there is a cycle in G passing through all members of S (in any order). The maximum value of k for which a graph G has property $C(k)$ is called the *cyclability* of G. This concept was first introduced 1971 by Chvátal (cf. [7]). It is an old, and well-known, result of Dirac [8] that every k-connected graph possesses property $C(k)$. If one assumes that the graphs under consideration have more restrictive properties, one can often do better than Dirac's result. For example, if a graph G is k-connected and k-regular, then G has property $C(k+4)$ (cf. [11,13]), if it is 3-connected and cubic, it has property $C(9)$ (cf. [14]), if it is 3-connected, planar and not K_4, it has property $C(5)$ (cf. [18]) and, finally, if it is 3-connected, cubic and planar, it is known to have property $C(23)$ (cf. [2]). In the last three of these four examples, the bound expressed is sharp.

Thomas and Yu [24,25] proved that every 4-connected projective-planar graph is Hamiltonian, thus proving a conjecture of Grünbaum. Altschuler [3] showed that every 6-connected toroidal graph is Hamiltonian. Brunet and Richter [6] showed that every 5-connected triangulation of the torus is Hamiltonian. (See also [25].) A stronger result was conjectured independently by Grünbaum [10] and Nash-Williams [16], namely, that every 4-connected toroidal graph is Hamiltonian, but this remains open. For the Klein bottle, Brunet, Nakamoto and Negami [5] proved that every 5-connected triangulation is Hamiltonian. Kawarabayashi [12] has conjectured more, namely, that every 4-connected graph embeddable on the Klein bottle is Hamiltonian, but to date this too is unproved.

Of course, any Hamiltonian graph G is, in particular, $C(|V(G)|)$. There are a host of papers on cyclability for special classes of graphs, but they are too numerous to mention here.

A graph G is said to have property $P(2, k)$ if given any set of $k+2$ vertices $\{a, b; v_1, \dots, v_k\}$ there is a path in G joining vertices a and b and passing through all of $v_1, \dots, v_k$ (in any order). It follows easily from a result of Dirac (again cf. [8]) that

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if a graph G is n-connected for $n \geq 2$, then G is $P(2, n-1)$. (See [17].) If G is 4-connected and planar, it is Hamiltonian-connected, by a beautiful result of Thomassen [20] and hence is $P(2, |V(G)| - 2)$. On the other hand, if a graph G is only 3-connected, it is $P(2, 3)$, but there are infinitely many 3-connected (and even planar) graphs which are not $P(2, 4)$. (Cf. [17].)

A *surface* Σ is a connected compact Hausdorff space which is locally homeomorphic to an open disc in the plane. If a graph G is embedded in the surface Σ, we denote by $eg(G, \Sigma)$ the number satisfying the equation $|V(G)| - |E(G)| + |F(G)| = 2 - eg(G, \Sigma)$ where $V(G)$, $E(G)$ and $F(G)$ denote the sets of vertices, edges and faces of the embedded graph respectively. ($eg(G, \Sigma)$ is called the *Euler genus* of the embedding.)

The *representativity* (or *face-width*) of a graph G embedded in a surface Σ, denoted $fw(G, \Sigma)$, is defined as the $\min\{|\Gamma \cap V(G)| : \Gamma$ is a homotopically nontrivial simple closed curve in Σ and $\Gamma \cap G \subseteq V(G)\}$. Let x and y be vertices in G. We use $d_{G\Sigma}(x, y)$ to denote the $\min\{|\Gamma \cap V(G)| : \Gamma$ is a simple curve in Σ from x to y and $\Gamma \cap G \subseteq V(G)\}$. Similarly, for two disjoint vertex sets C and D in G, we let $d_{G\Sigma}(C, D) = \min\{d_{G\Sigma}(x, y) : x \in C$ and $y \in D\}$. For a comprehensive treatment of representativity as well as embedded graphs in general, we refer the reader to [15].

Thomassen [22] conjectured that if the representativity of a 5-connected triangulation is large enough, then it is Hamiltonian. This was subsequently proved by Yu [27]. Later, the first author [12] showed more; namely, that every 5-connected triangulation of a surface of large enough representativity is, in fact, Hamiltonian-connected.

Thomassen [22] also pointed out that 5-connectivity is best possible here by providing for every surface constructions of 4-connected graphs embedded with arbitrarily large representativity which are not even 1-tough (and hence not Hamiltonian).

In the present paper, we will use the so-called method of "planarizing cycles", introduced in [21,23] and refined further in [27] to show that a 4-connected graph embedded in an orientable surface with sufficiently large representativity (as a function of the genus and the integer k), has property $P(2, k)$ (respectively, property $C(k)$). To accomplish this, we will make use of the following theorem of Yu. (For an application of this theorem to matching by Aldred and the present authors, see [1]).

Theorem 1.1 ([27]). *Let G be a connected graph embedded in a surface Σ (other than the sphere) with $eg(G, \Sigma) = g$ and $fw(G, \Sigma) \geq 8(d+1)(2^g - 1)$. Then G can be reduced to a graph H embedded in a disjoint union S of spheres by cutting along a set of "planarizing" cycles $\{C_1, \ldots, C_m\}$ (in this order) such that*

(i) *each C_i is induced,*

(ii) *for every integer k with $0 \leq k \leq d/2$ there is an induced cycle $D_i^{k'}$ (and D_i^k if C_i is orientation preserving) which bounds a closed disc in S containing C_i' (and C_i) such that for every vertex $z \in D_i^{k'}$ (and $z \in D_i^k$) there is a simple curve P in S from z to C_i' (and C_i) with length equal to $d_{HS}(z, C_i') = k+1$ (and $d_{HS}(z, C_i) = k+1$) and $P \cap D_i^{k'} = \{z\}$ (and $P \cap D_i^k = \{z\}$), and*

(iii) *all D_i^k and $D_i^{k'}$ are disjoint, and for each integer k with $0 \leq k \leq d/2$, the closed disc bounded by $D_i^{k'}$ containing C_i' is disjoint from the closed disc bounded by D_i^k containing C_i, and both do not contain the disc bounded by $D_j^{k'}$ or D_j^k containing C_j' or C_j for any $j > i$.*

It was subsequently pointed out, first for triangulations in [21] and later for arbitrary 2-connected graphs in [4], that if one starts with a 2-connected graph G in the above theorem, then the planarizing cycles can be chosen in such a way that when the vertices and edges of G lying in the interiors of the g cylinders between $D_i^{k'}$ and D_i^k, for $1 \leq i \leq g$, are deleted, the resulting plane graph H is connected.

We shall also use a result of Sanders [19] on paths in planar graphs. We follow the terminology and notation of his paper to introduce this result. For a subgraph H of G, the *bridges* of H in G are defined as follows. A *trivial bridge* of H in G is an edge in $E(G) \setminus E(H)$ with both ends in $V(H)$. A *non-trivial bridge* of H in G is a component K of $G \setminus H$ with all vertices of H adjacent to vertices of K added and all edges with one end in H and the other in K added. The *vertices of attachment* of a bridge B of H in G is the set $V(B) \cap V(H)$. A bridge is *attached to* its vertices of attachment. A path (cycle) P, a subgraph of a plane G, is a *Tutte path* (*cycle*) if and only if each bridge of P has at most three vertices of attachment and each bridge containing an edge of the infinite face boundary has at most two vertices of attachment. Sanders' theorem can then be stated as follows.

Theorem 1.2 ([19]). *Let G be a 2-connected plane graph. Let e be an edge of the boundary of the infinite face, and let x and y be arbitrary distinct vertices of G. Then G has a Tutte path P from x to y through edge e.*

2. The main results

Theorem 2.1. *Suppose G is a 4-connected graph embedded in an orientable surface Σ with Euler genus $eg(G, \Sigma) = g$ and that k is a positive integer. Then if $fw(G) > f(g, k) = 8(4k+13)(2^g - 1)$ and if $\{a, b; v_1, \ldots, v_k\}$ is any set of $k+2$ distinct vertices, there is a path P joining a and b which passes through all of $v_1, \ldots, v_k$.*

Proof. Let $S = \{a, b; v_1, \ldots, v_k\}$ be a set of $k+2$ distinct vertices. If $g = 0$, i.e., G is planar, then G is Hamiltonian-connected (i.e., it has a Hamiltonian path joining every pair of vertices) by Corollary 2 of [20] and we are done. So henceforth we shall assume that $g > 0$.

First note that if one slices a cylinder via at least $4k + 12$ slices, $C_1, \ldots, C_{4k+12}$, there must be at least four consecutive slices in this sequence such that none of the subcylinders thus formed or their boundaries contain any vertex of S. So in the