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The complexity of locally injective homomorphisms

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Dedicated to Carsten Thomassen on the occasion of his 60th birthday

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ABSTRACT

A homomorphism $f : G \to H$, from a digraph G to a digraph H, is locally injective if the restriction of f to $N^-(v)$ is an injective mapping, for each $v \in V(G)$. The problem of deciding whether such an f exists is known as the injective H-colouring problem (INJ-HOM$_H$). In this paper, we classify the problem INJ-HOM$_H$ as being either a problem in P or a problem that is NP-complete. This is done in the case where H is a reflexive digraph (i.e. H has a loop at every vertex) and in the case where H is an irreflexive tournament. A full classification in the irreflexive case seems hard, and we provide some evidence as to why this may be the case.

1. Introduction

Let G and H be digraphs. The homomorphism $f : G \to H$ is said to be *locally injective on in-neighbours* if $f|_{N^-(v)}$ is an injective mapping from $V(G)$ to $V(H)$ for every $v \in V(G)$. Therefore, the in-neighbours of every vertex $v \in V(G)$ have to be mapped to distinct vertices in H while preserving the arcs of G.

The problem of deciding whether such an f exists (for a fixed H) is known as the locally injective (on in-neighbours) H-colouring problem, and is denoted by INJ-HOM$_H$.

Problem 1.1 INJ — HOM$_H$

Instance: A digraph G.
Question: Is there a locally injective (on in-neighbours) homomorphism $f : G \to H$?

One can of course also define a homomorphism $f : G \to H$ to be injective if it is injective on the out-neighbours of the vertices in G. This would be the same as requiring the homomorphism to be injective on the in-neighbours of the converse of G (where of course we are now also mapping to the converse of H).

It is also possible to insist that the homomorphism be injective on $N^+(v) \cup N^-(v)$ for every $v \in V(G)$. This is exactly what happens in the undirected case. If G and H are undirected graphs, then the homomorphism $f : G \to H$ is said to be injective if f is injective on the neighbourhood of every vertex in G.

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Locally injective homomorphisms of undirected graphs are studied in [6–10]. In some cases they are referred to as *partial covers*, in contrast to full covers, which may be viewed as locally bijective homomorphisms (the mapping is bijective on neighbourhoods). If the target in the locally injective homomorphism problem is a complete undirected graph, one obtains the *injective chromatic number*. This is the smallest n such that a given graph G has a locally injective homomorphism to K_n.

If the target in the locally injective homomorphism problem is a reflexive complete undirected graph, one obtains the *reflexive injective chromatic number*. This may be viewed as the least number of colours needed to colour the vertices of a graph such that the neighbours of a vertex all receive different colours, with the added condition that adjacent vertices may receive the same colour. Hahn et al. [10] were particularly interested in determining the reflexive injective chromatic number of the hypercube because of its connections to coding theory. Since we are mostly interested in complexity results, this will be the only type of result from [10] that we will list here. The reflexive injective chromatic number problem (RICN_k) may be stated formally as follows.

Problem 1.2 RICN_k

Instance:	A graph G and a natural number k.
Question:	Is there a reflexive injective k-colouring of G?

Theorem 1.1 (*[10]*). *The problem* RICN_k *is NP-complete for every fixed* $k \geq 3$.

Fiala and Kratochvíl [6–8] and Fiala et al. [9] consider the more general problem of locally injective homomorphisms where the target is not a complete graph. In all of these papers it is pointed out that the locally injective homomorphism problem is connected to so-called $L(2, 1)$ labellings of graphs: adjacent vertices must receive labels that differ by at least two, while vertices at distance two must receive labels that differ by at least one. This has applications in radio frequency assignment. Radio towers (for example cell phone towers) that are close together (where interference is quite possible) need frequencies that are far apart. Towers that are not as close to each other may only need frequencies that differ by a smaller amount.

The complexity results in these papers are mostly centered around a family of graphs which we describe next. Denote by $\Theta(a_1, a_2, \ldots, a_n)$ the (multi)graph that is formed by joining two vertices by n internally disjoint paths of lengths $a_1, a_2, \ldots, a_n$. An abbreviated version of this notation is as follows: $\Theta(a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n})$ is taken to mean that there are k_i paths of length a_i joining the two vertices.

The authors show that the family of graphs defined above exhibit both problems that are polynomial and problems that are NP-complete:

- $\Theta(a^n)$ is polynomial for every fixed a.
- $\Theta(a^i, b^j)$ is polynomial for every odd a, b.
- $\Theta(a^i, b^j)$ is NP-complete for every a and b of different parity.
- $\Theta(a, b, c)$ is NP-complete if c is divisible by $a + b$.
- $\Theta(1, 2, c)$ is NP-complete for every c.
- $\Theta(a^i, b^j)$ is polynomial when a and b are divisible by the same power of 2 or if $i + j \leq 2$. It is NP-complete otherwise.
- $\Theta(1, 2, a)$ and $\Theta(1, 3, b)$ are NP-complete for all positive integers $a > 2$ and $b > 3$.
- For any three distinct odd positive integers a, b and c, $\Theta(a, b, c)$ is NP-complete.

As with ordinary homomorphisms, the hope is that the injective problems will exhibit a dichotomy. Towards this end Fiala and Kratochvíl [8] considered the list version of the locally injective homomorphism problem.

The list version of the injective homomorphism problem with target H is the problem INJ-LIST-HOM_H shown below.

Problem 1.3 INJ-LIST-HOM_H

Instance:	A graph G and lists $L(v) \subseteq V(H)$.
Question:	Does there exist a locally injective homomorphism $f : G \to H$ such that $f(v) \in L(v)$ for every $v \in V(G)$?

The lists $L(v)$ are to be thought of as admissible images for the vertex $v \in V(G)$. Fiala and Kratochvíl [8] were able to show that for this problem there is a dichotomy.

Theorem 1.2 (*[8]*). *The list, injective, homomorphism problem with target H is solvable in linear time if the graph H contains at most one cycle in each component. It is NP-complete otherwise.*