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Cyclability in k-connected $K_{1,4}$-free graphs

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ABSTRACT

It is proved that if G is a k-connected $K_{1,4}$-free graph and S is a subset of vertices such that $k \geq 3$ and $|S| \leq 2k$ then G has a cycle containing S. A similar result is obtained when restricting the k-connectivity assumption to the subset S.

1. Introduction

Various problems on cycles are studied in graph theory. A classic one concerns the circumference of undirected graphs, in relation with the Hamiltonian problem. The circumference of a graph G is the length of a longest cycle in G, denoted by $c(G)$. In this paper we study the more general notion of cyclability of graphs, i.e. we look for cycles containing as many prescribed vertices as possible. The *cyclability* $C(G)$ of a graph G is defined to be the largest integer m such that G has a cycle through any m vertices. (Notice that $C(G) = n = |V(G)|$ is the case of Hamilton cycles.)

The basic and classic results in the field are due to Dirac.

Theorem 1 ([3]). *If G is a 2-connected graph on $n \geq 3$ vertices with minimum degree δ, then $c(G) \geq \min\{n, 2\delta\}$.*

Theorem 2 ([2]). *If G is a k-connected graph then it has a cycle through any k vertices, i.e. $C(G) \geq k$.*

Flandrin et al. [4] have generalized Theorem 2 by limiting the connectivity condition of the graph into a connectivity condition on a subset of vertices considered. Let G be an arbitrary graph and $S \subset V(G)$ be a set of at least two vertices. S is k-connected in G if any two vertices of S can not be separated in G by deleting at most $k - 1$ vertices.

Theorem 3 ([4]). *If S is a k-connected subset of vertices in a graph G, then there is a cycle through any k vertices of S.*

These theorems are sharp (see examples: the bipartite complete graphs $K_{k,k+1}$). However, it turned out that if the graph does not contain any *claw*, i.e. an induced subgraph isomorphic to $K_{1,3}$, then much stronger results can be proved. Let us recall the two well-known equivalent conjectures due to Thomassen [8] and to Matthews and Sumner [7].

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Conjecture 1 ([8]). *Every 4-connected line graph is Hamiltonian.*

Conjecture 2 ([7]). *Every 4-connected claw-free graph is Hamiltonian.*

Li [6] proved the second conjecture for any 6-connected claw-free graph with at most 33 vertices of degree 6. On the other hand, if you are interested in cyclability of the graph, Győri and Plummer [5] proved that 3-connected claw-free graphs have a cycle through any 9 vertices and this is the best possible.

In this paper, we further weaken the conditions and assume only $K_{1,4}$-freeness instead of claw-freeness. We also limit the connectivity condition on subsets of vertices considered. Our main theorem is as follows:

Theorem 4. *Let G be a $K_{1,4}$-free graph and S be a k-connected subset of vertices in G with $k \geq 3$ and $|S| \leq 2k$. Then there exists a cycle containing S.*

Corollary 5. *Let G be a k-connected $K_{1,4}$-free graph, $k \geq 3$, then $C(G) \geq 2k$.*

Consider the graph $K_{2,3}$ and let S be the vertices in the three vertex partite. Then S is 2-connected and there is no cycle containing S. Therefore the condition $k \geq 3$ is necessary in Theorem 4 and Corollary 5. We think that the bound $2k$ is not sharp. But the Peterson graph shows that this bound is not larger than $3k$.

2. Proof of Theorem 4

We first give some useful definitions and notations.

All the graphs considered in this paper are undirected and simple. We use the notation and terminology in [1]

Given a graph G and $S \subset V(G)$, the vertices of the subset S are called S-vertices.

If $C = c_1c_2\dots c_pc_1$ is a cycle, we let $C[c_i, c_j]$ be the subpath $c_ic_{i+1}\dots c_j$, and $\overline{C}[c_j, c_i] = c_jc_{j-1}\dots c_i$, where the indices are taken modulo p. We will consider $C[c_i, c_j]$ and $\overline{C}[c_j, c_i]$ both as paths and as vertex-sets. Define $C(c_i, c_j] = C[c_{i+1}, c_j]$, $C[c_i, c_j) = C[c_i, c_{j-1}]$ and $C(c_i, c_j) = C[c_{i+1}, c_{j-1}]$. For any i, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, and for any $j \geq 2$, $c_i^{+j} = c_{i+j}$ and $c_i^{-j} = c_{i-j}$. For $A \subseteq C$, we set $A^+ = \{v^+ | v \in A\}$, $A^- = \{v^- | v \in A\}$, for any $j \geq 2$, $A^{+j} = \{v^{+j} | v \in A\}$ and $A^{-j} = \{v^{-j} | v \in A\}$. We use similar definitions for a path.

We now give the proof of Theorem 4.

Let $C = c_1c_2c_3\dots c_pc_1$ be a cycle in G such that

(a) $|S \cap C|$ is as large as possible and
(b) subject to (a), C is as long as possible.

By Theorem 3, we may assume that $|S \cap V(C)| \geq k$. It is proved in [4] and an easy consequence of Menger's Theorem that if S is k-connected then there are k paths between any vertex x and any k vertices in $S - \{x\}$, such that the paths are vertex disjoint except at x.

Let w be a vertex in $S - C$ and R the connected component in $G - V(C)$ containing w. S is k-connected and hence there are h, $h \geq k$, internally disjoint paths $H_i[w, x_i]$, $1 \leq i \leq h$, between w and h distinct vertices $x_1, x_2, \dots, x_h \in V(C)$. Notice that we assume for every i, $1 \leq i \leq h$, $H_i[w, x_i] \bigcap C = \{x_i\}$. Let $X = \{x_1, x_2, \dots, x_h\}$. The vertices in X divide the cycle C into h segments $C(x_i, x_{i+1})$, $1 \leq i \leq h$ with $x_{h+1} = x_1$.

It follows that every segment $C(x_i, x_{i+1})$, $1 \leq i \leq h$, contains at least one vertex in S, since if $C(x_i, x_{i+1}) \cap S = \emptyset$, the new cycle $C[x_{i+1}, x_i]\overline{H}_i[x_i, w]H_{i+1}(w, x_{i+1}]$ contradicts (a). If $|C(x_i, x_{i+1}) \cap S| \geq 2$ for all $1 \leq i \leq h$, then $|C \cap S| \geq 2k$ and we are done. So there exists some segment containing exactly one S-vertex. Let $Y = \{C(x_i, x_{i+1}) : 1 \leq i \leq h, |C(x_i, x_{i+1}) \cap S| = 1\}$ and $I = \{i : 1 \leq i \leq h, C(x_i, x_{i+1}) \in Y\}$. For any i in I, put $y_i = C(x_i, x_{i+1}) \bigcap S$ and let $Y^0 = \{y_i : i \in I\}$.

We will use the following lemma and claims:

Lemma 1. *Let G be a graph, $M = \{u_1, u_2, \dots\dots, u_k\} \subset V(G)$ and v, x_1, x_2 three distinct vertices with $v \notin M$. Suppose that there are two internally disjoint paths $Q_j[v, x_j]$, $1 \leq j \leq 2$ from v to x_1 and x_2 and there are k internally disjoint paths $T_i[v, u_i]$, $1 \leq i \leq k$. Then there are k paths $P_i[v, w_i]$, $1 \leq i \leq k$ from v to k distinct vertices $\{w_i : 1 \leq i \leq k\} \subseteq M \bigcup \{x_1, x_2\}$, with $w_{k-1} = x_1$ and $w_k = x_2$, such that $\bigcup_{i=1}^{k} V(P_i[v, w_i]) \subseteq (\bigcup_{i=1}^{k} V(T_i[v, u_i])) \bigcup (\bigcup_{j=1}^{2} V(Q_j[v, x_j]))$ and the paths $P_i[v, w_i]$, $1 \leq i \leq k$, are internally disjoint.*

Proof of Lemma 1. If one of x_1 and x_2, say $x_1 \notin \{u_i : 1 \leq i \leq k\}$, denote by a the last vertex in $Q_1[v, x_1]$ that is also in $\bigcup_{j=1}^{k} T_i[v, u_i]$, say $a \in T_k[v, u_k]$ we put $Z_1[v, x_1] = T_k[v, a)Q_1[a, x_1]$. Then we get k disjoint (except at v) paths $T_i[v, u_i]$, $1 \leq i \leq k-1$ and $Z_1[v, x_1] = T_k[v, a)Q_1[a, x_1]$ with the property that $u_i \in M$, $1 \leq j \leq k-1$ and Z_1 has one end in $\{x_1, x_2\}$. Without loss of generality we assume that $|T_k[v, a]|$ is as small as possible.

If $Q_2(v, x_2] \cap ((\bigcup_{i}^{k-1} T_i(v, u_i]) \bigcup Z_1(v, x_1]) = \emptyset$, then $P_i[v, w_i] := T_i[v, u_i]$, $1 \leq i \leq k-2$, $P_{k-1}[v, w_{k-1}] := Z_1[v, x_1]$ and $P_k[v, w_k] := Q_2[v, x_2]$ satisfy our lemma.