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Maximum independent sets in 3- and 4-regular Hamiltonian graphs

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ABSTRACT

Smooth 4-regular Hamiltonian graphs are generalizations of cycle-plus-triangles graphs. While the latter have been shown to be 3-choosable, 3-colorability of the former is NP-complete. In this paper we first show that the independent set problem for 3-regular Hamiltonian planar graphs is NP-complete, and using this result we show that this problem is also NP-complete for smooth 4-regular Hamiltonian graphs. We also show that this problem remains NP-complete if we restrict the problem to the existence of large independent sets (i.e., independent sets whose size is at least one third of the order of the graphs).

1. Introduction

In an earlier paper [1], two of the present authors considered the question of the 3-colorability of the so-called smooth 4-regular Hamiltonian graphs. These are generalizations of the well-known cycle-plus-triangles graphs; for the latter, 3-colorability had been conjectured by Erdös [5] and was proved by Fleischner and Stiebitz [2]. A 4-regular graph G with a Hamiltonian cycle H is smooth if each component of the edge-complement $G \setminus H$ is a cycle which is "non-selfcrossing" in the sense that the cyclic order of its vertices agrees with their cyclic order of H.

It was shown in [1] that the problem of deciding whether an arbitrary smooth 4-regular Hamiltonian graph is 3-colorable is NP-complete. As 3-colorability of a graph of order n implies the existence of a "large" independent set of vertices, i.e. having cardinality at least $n/3$, the result of [1] suggests that the problem of deciding the existence of such a set for an arbitrary smooth 4-regular Hamiltonian graph also may be NP-complete. In the present paper we show that this is indeed the case.

In general, two problems that are similar in spirit present themselves.

Large independent set problem (LIS): Given a graph G of order n and maximum degree d, does G contain an independent set of vertices of cardinality at least $n/(d-1)$?

Note that by Brooks' theorem any graph of maximum degree $d \geq 4$ except K_{d+1} has an independent set of cardinality $\geq n/d$. Independent sets of cardinality $\geq n/(d-1)$ will be called *large*; for the 4-regular graphs under consideration in the present paper this means independent sets of size at least $n/3$.

Maximum independent set problem (MIS): Given a graph G and a positive integer k, does G contain an independent set of vertices of cardinality at least k?

We consider the two problems for the class $\mathbf{S}$ of all smooth 4-regular Hamiltonian graphs. Our main result is that both MIS and LIS are NP-complete in this class (Theorems 3.3 and 5.2).

Starting with the well-known fact that MIS is NP-complete in the class $\mathbf{G}_3$ of all 3-regular graphs we show first that MIS is also NP-complete in $\mathbf{H}_3$, the class of all Hamiltonian 3-regular graphs (Section 2). The proof of the NP-completeness of MIS

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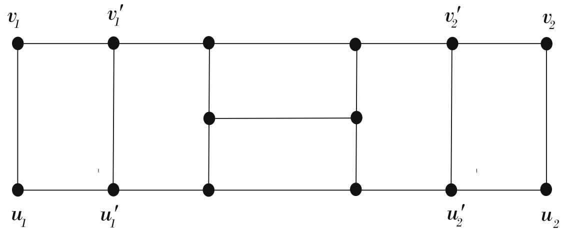

Fig. 1. The graph L.

in **S** is then carried out by a polynomial reduction from **S** to $\mathbf{H}_3$ (Sections 3 and 4). It is a special feature of all the reductions and constructions carried out in this paper that they produce planar graphs when applied to planar graphs to begin with.

The graphs we consider are finite and simple. We note, however, that our results on MIS and LIS also hold for multigraphs provided one edge of every pair of parallel edges belongs to H. An edge e with endpoints x, y will be denoted by $e = xy$. If H is a subgraph of a graph G, we denote by $G \setminus H$ the edge-complement of H in G, i.e. the graph with $V(G \setminus H) = V(G)$, $E(G \setminus H) = E(G) \setminus E(H)$. Thus $G \setminus H$ may have isolated vertices.

By a *Hamiltonian graph* we mean a pair (G, H), where G is a graph and H a Hamiltonian cycle of G. Occasionally we refer to G alone as being a Hamiltonian graph, but it is always understood that a specific Hamiltonian cycle is given. If G is 4-regular the components of $G \setminus H$ are called the *inner cycles* of G.

The following concept which refers to both cycles and paths will be needed. Let D_1, D_2 be subgraphs of a graph G such that $V(D_2) \subset V(D_1)$, and D_i is either a cycle or a path whose endpoints are non-adjacent in G, $i = 1, 2$. If D_i is a path, let C_i be the cycle obtained from D_i by adding an edge joining its endpoints; if D_i is a cycle, let $C_i = D_i$. We say that D_2 is *non-selfcrossing* with respect to D_1, if the cyclic order of the vertices of C_2 coincides with their cyclic order on C_1, i.e. if the cyclic order of the vertices of C_2 is the restriction to $V(D_2)$ of the cyclic order on C_1.

As already mentioned at the beginning of this section, a 4-regular Hamiltonian graph (G, H) will be called *smooth* if its inner cycles are non-selfcrossing with respect to H.

The *independence number* of a graph G, denoted by α_G, is the cardinality of a maximum independent set of vertices of G. The *independence ratio* of G is α_G/n, where n is the order of G.

2. 3-regular Hamiltonian graphs

That MIS is an NP-complete problem for 3-regular Hamiltonian graphs is probably a well-known piece of graph-theoretic folklore. Not having found any reference in the literature, we include here a proof, at the same time strengthening the result by showing that NP-completeness of MIS already holds for *planar* cubic Hamiltonian graphs.

Proposition 2.1. MIS *is NP-complete in the class* $\mathbf{PH}_3$ *of all planar* 3*-regular Hamiltonian graphs.*

This follows at once from the fact that MIS is NP-complete in the class $\mathbf{P}_3$ of all planar 3-regular graphs [3, p. 194] and Lemmas 2.2 and 2.3.

Let L be the ladder-like graph on 14 vertices shown in Fig. 1. One easily checks that its independence number is 6, and that each maximum independent set of L contains exactly one of u_1, v_1 and exactly one of u_2, v_2.

Consider an arbitrary (not necessarily connected) graph G of order n having two non-adjacent edges e_1, e_2 and subdivide each by two new vertices. Denote by e_i' the new edge joining the two subdivision vertices of e_i, $i = 1, 2$. Form G' by taking a copy of L disjoint from G and identifying the endpoints of e_i' with u_i, v_i, $i = 1, 2$. It does not matter which endpoint of e_i' is identified with u_i and which with v_i. The operation just described of adding the graph L to G will be referred to as an *L-insertion*. Obviously, L-insertion preserves 3-regularity.

Lemma 2.2. *If* G' *is obtained form* G *by an* L*-insertion, then* $\alpha_{G'} = \alpha_G + 6$.

Proof. It follows immediately from the properties of L that, if J is an independent set of G, then the union of J and a suitably chosen maximum independent set of L is an independent set of G' of cardinality $|J| + 6$. Hence

$$\alpha_{G'} \geq \alpha_G + 6.$$

Conversely, given an independent set I_0 of G', consider $J_0 = I_0 \cap V(G)$. There are two possibilities.

Case 1. J_0 is an independent set of G. Then $I_0 \setminus J_0$ is an independent set of L; hence $|I_0 \setminus J_0| \leq 6$, and therefore $|I_0| \leq |J_0| + 6$.

Case 2. J_0 contains two vertices x, y such that $xy \in E(G)$. Then xy is one of the edges used in carrying out the L-insertion, say $xy = e_1$. It follows that, in G', one of u_1, v_1 is a neighbor of x, and the other a neighbor of y, say $xu_1, yv_1 \in E(G')$. Therefore, $u_1, v_1 \notin I_0$. Now define I_1 as follows:

$$I_1 = \begin{cases} (I_0 \setminus \{y\}) \cup \{v_1\}, & \text{if } u_1' \in I_0, \\ (I_0 \setminus \{x\}) \cup \{u_1\}, & \text{if } v_1' \in I_0, \end{cases}$$