Discrete Mathematics 310 (2010) 2965–2973

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Injective colorings of sparse graphs

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ARTICLE INFO

Article history:
Received 22 August 2009
Received in revised form 2 July 2010
Accepted 5 July 2010
Available online 27 July 2010

Keywords:
Injective coloring
Maximum average degree
Planar graph

ABSTRACT

Let $\text{mad}(G)$ denote the maximum average degree (over all subgraphs) of G and let $\chi_i(G)$ denote the injective chromatic number of G. We prove that if $\text{mad}(G) \leq \frac{5}{2}$, then $\chi_i(G) \leq \Delta(G) + 1$; and if $\text{mad}(G) < \frac{42}{19}$, then $\chi_i(G) = \Delta(G)$. Suppose that G is a planar graph with girth $g(G)$ and $\Delta(G) \geq 4$. We prove that if $g(G) \geq 9$, then $\chi_i(G) \leq \Delta(G) + 1$; similarly, if $g(G) \geq 13$, then $\chi_i(G) = \Delta(G)$.

1. Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring. Injective colorings have their origin in complexity theory [13], and can be used in coding theory.

Note that an injective coloring is not necessarily proper, and in fact, $\chi_i(G) = \chi(G^{(2)})$, where the *neighboring graph* $G^{(2)}$ is defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. It is clear that $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of graph G. Graphs attaining the upper bound were characterized in [13], and it was also shown that for every fixed $k \geq 3$ the problem of determining if a graph is injective k-colorable is NP-complete.

As one can see, injective coloring is a close relative of the coloring of square of graphs and of $L(2, 1)$-labeling, which have both been studied extensively. Upper bounds on $\chi(G^2)$ and on the $L(2, 1)$-labeling number $\lambda(G)$ are both upper bounds on $\chi_i(G)$.

Alon and Mohar [1] showed that if G has girth at least 7, then $\chi(G^2)$ could be as large as $\frac{c\Delta^2}{\log \Delta}$ (for some constant c), but not larger. This gives an upper bound on $\chi_i(G)$ when graph G has girth at least 7. The study of $\chi(G^2)$ has been largely focused on the well-known Wenger's Conjecture [16].

Conjecture 1 (*Wenger [16]*). *If G is a planar graph with maximum degree Δ, then $\chi(G^2) \leq 7$ when $\Delta = 3$, $\chi(G^2) \leq \Delta + 5$ when $4 \leq \Delta \leq 7$, and $\chi(G^2) \leq 3\Delta/2 + 1$ when $\Delta \geq 8$.*

This conjecture and its variations have been studied extensively; see [4] or [10] for a good survey.

Much effort has been spent on finding graphs with low injective chromatic numbers, namely graphs with $\chi_i(G) \leq \Delta + c$, for some small constant c. In Theorems 2 and 3, we list some of the most recent related results.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.07.003

Theorem 2. *Let G be a planar graph with maximum degree $\Delta(G) \geq D$ and girth $g(G) \geq g$. Then*

(a) [5] *if* $(D, g) \in \{(3, 24), (4, 15), (5, 13), (6, 12), (7, 11), (9, 10), (15, 8), (30, 7)\}$, *then* $\chi_i(G) \leq \chi(G^2) = \Delta + 1$.
(b) [4] $\chi_i(G) \leq \chi(G^2) \leq \Delta + 2$ *if* $(D, g) = (36, 6)$.
(c) [14] $\chi_i(G) \leq \Delta + 4$ *if* $g = 5$; $\chi_i(G) \leq \Delta + 1$ *if* $g = 10$; *and* $\chi_i(G) = \Delta$ *if* $g = 19$.
(d) [7] $\chi_i(G) \leq \Delta + 2$ *if* $g = 8$; $\chi_i(G) \leq \Delta + 1$ *if* $g = 11$; *and* $\chi_i(G) = \Delta$ *if* $(D, g) \in \{(3, 20), (71, 7)\}$.

Instead of studying planar graphs with high girth, some researchers consider graphs with bounded *maximum average degree*, $\text{mad}(G)$, where the average is taken over all subgraphs of G. Note that every planar graph G with girth at least g satisfies $\text{mad}(G) < \frac{2g}{g-2}$. Below are some results in terms of $\text{mad}(G)$.

Theorem 3. *Let G be a graph with maximum degree $\Delta(G) \geq D$ and $\text{mad}(G) < m$. Then*

(a) [9] $\chi_i(G) \leq \Delta + 8$ *if* $m = 10/3$; $\chi_i(G) \leq \Delta + 4$ *if* $m = 3$; *and* $\chi_i(G) \leq \Delta + 3$ *if* $m = 14/5$.
(b) [8] $\chi_i(G) \leq \Delta + 2$ *if* $(D, m) = (4, 14/5)$; $\chi_i(G) \leq 5$ *if* $(D, m) = (3, 36/13)$ *and* $m = 36/13$ *is sharp.*

The major tool used in the proofs of Theorems 2 and 3 is the discharging method, which relies heavily on the idea of reducible subgraphs. A *reducible subgraph H* is a subgraph such that any coloring of $G - H$ can be extended to a coloring of G. Since the coloring of $G - H$ will restrict the choice of colors on H, these arguments work well when the graph G is sparse.

In particular, if $\text{mad}(G)$ is much smaller than $\Delta(G)$, then we are guaranteed a vertex v with degree much smaller than $\Delta(G)$. Such a vertex v is a natural candidate to be included in our reducible subgraph H, since v has at least $\Delta(G)$ allowable colors and only has a few restrictions on its color. However, if $\text{mad}(G)$ is nearly as large as $\Delta(G)$, then we are not guaranteed the presence of such a low degree vertex v. Here it is less clear how to proceed. Thus, proving results when $\Delta(G) - \text{mad}(G)$ is small is a much harder task than proving analogous results when $\Delta(G) - \text{mad}(G)$ is larger.

In this paper, we study graphs with low injective chromatic number, namely G such that $\chi_i(G) \leq \Delta + 1$. We consider sparse graphs with bounded maximum average degree, which include planar graphs with high girth. Our results below extend or generalize the corresponding results in Theorem 2.

Theorem 4. *Let G be a graph with maximum degree Δ.*

(a) *If* $\text{mad}(G) \leq \frac{5}{2}$, *then* $\chi_i(G) \leq \Delta + 1$.
(b) *If* $\text{mad}(G) < \frac{42}{19}$, *then* $\chi_i(G) = \Delta$.

Theorem 5. *Let G be a planar graph with girth $g(G)$ and maximum degree $\Delta \geq 4$.*

1. *If* $g(G) \geq 9$, *then* $\chi_i(G) \leq \Delta + 1$.
2. *If* $g(G) \geq 13$, *then* $\chi_i(G) = \Delta$.

Like many similar results, we use discharging arguments in our proofs. In most discharging arguments, the reducible subgraphs are of bounded size. Our contribution to injective coloring is using reducible configurations of arbitrary size, similar to the 2-alternating cycles introduced by Borodin [3] and generalized by Borodin, Kostochka, and Woodall [6].

Let G be a Class 2 graph, that is, suppose the edge-chromatic number of G is $\Delta + 1$. Let G' be the graph obtained from G by inserting a degree 2 vertex on each edge. Now $\chi_i(G') > \Delta$, for otherwise, we could color the edges of G by the colors of the 2-vertices in G' on the corresponding edges, which would give a Δ-edge-coloring of G. Here we list some Class 2 graphs; see [12] for more details.

Theorem 6 (*[12]*)**.**

(a) *If H is a regular graph with an even order, and G is a graph obtained from H by inserting a new vertex into one edge of H, then G is of Class* 2.
(b) *For any integers $D \geq 3$ and $g \geq 3$, there is a Class* 2 *graph of maximum degree D and girth g.*

By combining Theorem 6 with results on Class 2 graphs, we have the following corollary.

Corollary 7. *There are planar graphs with* $g(G) = 8$ *and* $\chi_i(G) \geq \Delta + 1$. *There are graphs with* $\text{mad}(G) = \frac{8}{3}$ *and* $\chi_i(G) \geq \Delta + 1$. *For any $\Delta \geq 3$ and $g \geq 3$, there are graphs with maximum degree Δ, girth $2g$, and* $\chi_i(G) \geq \Delta + 1$.

There are some gaps between the bounds in the above corollary and Theorems 4 and 5(a). Our bounds on mad and girth may be further improved if some clever idea is elaborated.

When we extend a coloring of $G - H$ to a subgraph H, the colors available for vertices of H are restricted, thus we will essentially supply a list of available colors for each vertex of H. The following two classic theorems on list-coloring will be used heavily.

Theorem A (*[15]*)**.** *Let L be an assignment such that $|L(v)| \geq d(v)$ for all v in a connected graph G.*

(a) *If $|L(y)| > d(y)$ for some vertex y, then G is L-colorable.*
(b) *If G is 2-connected and the lists are not all identical, then G is L-colorable.*