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Irreducible (v_3) configurations and graphs☆

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Abstract

Cubic bipartite graphs with girth at least 6 correspond to symmetric combinatorial (v_3) configurations. In 1887, Martinetti described a simple construction method which enables one to construct all combinatorial (v_3) configurations from a set of so-called *irreducible* configurations. The result has been cited several times since its publication, both in the sense of configurations and graphs. But after a careful examination, the list of irreducible configurations given by Martinetti has turned out to be incomplete. We will give the description of all irreducible configurations and corresponding graphs, including those which are missing in Martinetti's list.

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1. Introduction

Let us start with basic definitions. A (*combinatorial*) *configuration* (v_r, b_k) is an incidence structure of points and lines with the following properties:

(1) There are v points and b lines.
(2) There are r lines through each point and k points on each line.
(3) Two different points are connected by at most one line and two lines intersect in at most one point.

Note that configurations considered here are purely combinatorial objects and that there is no geometric significance associated with the terms point and line. For this reason, we will omit the adjective combinatorial and speak only of configurations. However, we will briefly discuss the geometric representation of configurations in the last section.

A (v_r, b_k) configuration is called *symmetric* if $v = b$ (which is equivalent to saying that $r = k$) and is denoted by (v_r) (in some papers also by v_r).

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Incidence structures and hence configurations are closely related to graphs. Let $G(\mathscr{C})$ be a bipartite graph with v *black* vertices representing points of the incidence structure $\mathscr{C}$, b *white* vertices representing lines of $\mathscr{C}$, and with an edge joining two vertices if and only if the corresponding point and line are incident in $\mathscr{C}$. We call $G(\mathscr{C})$ *incidence graph* or *Levi graph* or just *graph* of the incidence structure $\mathscr{C}$. The following proposition characterizes symmetric configurations in terms of their graphs.

Proposition 1. *An incidence structure is a (v_r) configuration if and only if its graph is r-regular with girth at least* 6.

For the proof and more about correlations between configurations and graphs see [6,9,10]. For enumeration results about (v_3) configurations the reader is referred to [2].

With each (v_r, b_k) configuration $\mathscr{C}$ the *dual* (b_k, v_r) configuration $\mathscr{C}^*$ may be associated by reversing the roles of points and lines in $\mathscr{C}$. Both $\mathscr{C}$ and $\mathscr{C}^*$ share the same incidence graph, only the black–white coloring of its vertices is reversed. If $\mathscr{C}$ is isomorphic to its dual we say that $\mathscr{C}$ is *self-dual* and the corresponding isomorphism is called a *duality*. A duality of order 2 is called a *polarity*. Configurations which admit a polarity are called *self-polar*.

If $P = \mathbb{Z}_v = \{0, 1, 2, \ldots, v-1\}$ represents a set of points and

$$\mathscr{B} = \{\{0, b, c\}, \{1, b+1, c+1\}, \ldots, \{v-1, b+v-1, c+v-1\}\}, \quad b, c \in P,$$

represents a set of lines of some (v_3) configuration $\mathscr{C}$ then $\mathscr{C}$ is called a *cyclic* (v_3) *configuration with base line* $\{0, b, c\}$. Of course, the idea can be generalized to cyclic (v_r) configurations for general values of r.

The Fano plane or the projective plane of order 2, the smallest (v_3) configuration, is the cyclic (7_3) configuration with base line $\{0, 1, 3\}$. Its incidence graph is the well-known Heawood graph. The second one in this family, the cyclic (8_3) configuration with base line $\{0, 1, 3\}$, is the only (8_3) configuration and is called the Möbius–Kantor configuration [6]. Let us mention also that incidence graphs of cyclic configurations correspond precisely to the so-called cyclic Haar graphs of girth at least 6, see [15].

In 1887, Martinetti suggested the following construction method for symmetric (v_3) configurations [16]. Suppose that in the given (v_3) configuration there exist two parallel (non-intersecting) lines $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ such that the points a_1 and b_1 are not on a common line. By removing these two lines, adding one new point c and three new lines $\{c, a_2, a_3\}$, $\{c, b_2, b_3\}$, $\{c, a_1, b_1\}$ we obtain a $((v+1)_3)$ configuration. It is not possible to obtain every (v_3) configuration from some $((v-1)_3)$ configuration by using this method. We will call (v_3) configurations which cannot be constructed in this way from a smaller one *irreducible configurations* and the others *reducible configurations*.

2. Irreducible graphs and configurations

In [16], Martinetti gave a list of irreducible configurations. He claimed that, in addition to some special cases for $v \leqslant 10$, there are two infinite families of irreducible (v_3) configurations. The result has been cited several times since its publication, both in the sense of configurations and graphs, for example in [1,5,8,13].

But in [13] the author expressed a certain amount of doubt about the result when saying: "The proof (of Martinetti's theorem) is, not surprisingly, involved and long; I have not checked the details, and I do not know it as a fact that anybody has. The statement has been accepted as true for these 112 years, and it may well be true. On the other hand, Daublebski's enumeration of the (12_3) configurations was also considered true for a comparable length of time…"

And indeed, after a careful examination, the list of irreducible configurations given by Martinetti has proved to be incomplete. The aim of this paper is to give the complete list of irreducible configurations and the corresponding graphs, including those which are missing in Martinetti's list.

To do this, we observe the Martinetti method on graphs of (v_3) configurations. For the sake of simplicity, we will use the notion (v_3) *graph* instead of graph of (v_3) configuration, i.e. (v_3) graph is a bipartite cubic graph with girth $\geqslant 6$. We define reducible and irreducible (v_3) graphs corresponding to reducible and irreducible configurations, respectively, as follows. A (v_3) graph G is *reducible* if there exists an edge $uv \in EG$ such that $(G - \{u, v\}) + x_1y_1 + x_2y_2$ or $(G - \{u, v\}) + x_1y_2 + x_2y_1$ is again a (v_3) graph, where x_1, x_2, y_1, y_2 are neighbors of u and v as it is shown in Fig. 1. Otherwise a (v_3) graph is *irreducible*.

In the proof of the Martinetti theorem, the following characterization of irreducible configurations will be useful.