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Discrete Mathematics 307 (2007) 345–351
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On cube-free median graphs

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Received 4 July 2003; received in revised form 17 July 2004; accepted 26 September 2004
Available online 28 August 2006

Abstract

Let G be a cube-free median graph. It is proved that $k/2 \geqslant \sqrt{n} - 1 \geqslant m/2\sqrt{n} \geqslant \sqrt{s} \geqslant r - 1$, where n, m, s, k, and r are the number of vertices, edges, squares, Θ-classes, and the number of edges in a smallest Θ-class of G, respectively. Moreover, the equalities characterize Cartesian products of two trees of the same order. The cube polynomial of cube-free median graphs is also considered and it is shown that planar cube-free median graphs can be recognized in linear time.

Keywords: Median graph; Cube-free graph; Cartesian product; Recognition algorithm

1. Introduction

Cube-free median graphs are, by definition, median graph without an induced 3-cube Q_3. This class of graphs naturally appears in different contexts. For instance, cube-free median graphs are precisely the bipartite absolute retracts without induced $K_{2,3}$ [3], and they play an important role in the location theory [2,11]. Cube-free median graphs are also precisely those median graphs for which the equality is attained in an Euler-type formula for median graphs [9].

Edges xy and uv of a graph G are in the Djoković–Winkler relation Θ [5,14] if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u).$$

Relation Θ is reflexive and symmetric in general and transitive on median graphs. Hence it partitions the edge set of a median graph into equivalence classes, called Θ-*classes*.

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1 Supported by the Ministry of Education, Science and Sport of Slovenia under the Grant Z1-3073-0101-01.
2 Supported by the same Ministry under the Grant J1-0504-0101.
3 Supported by the same Ministry under the Grant Z1-3219-0101.
4 Supported in part by the Ministry of Education of Czech republic, Project LN00A056.

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doi:10.1016/j.disc.2004.09.018

Let G be a cube-free median graph. Then the following invariants of G are important to us:

n the number of its vertices,
m the number of its edges,
s the number of its (induced) squares,
k the number of its Θ-classes, and
r the number of the edges in its smallest Θ-class.

The main result of this note asserts that

$$\frac{k}{2} \geqslant \sqrt{n} - 1 \geqslant \frac{m}{2\sqrt{n}} \geqslant \sqrt{s} \geqslant r - 1.$$

Moreover, if G is not a tree, then in any of the above inequalities, the equality holds if and only if G is the Cartesian product of two trees of the same order.

In the next section we recall concepts and results needed later. We follow this with a section in which the main result is proved. In the concluding section we give some more properties of cube-free median graphs. We give a few remarks on the cube polynomial of cube-free median graphs—we show that they always have two real zeros, and we give a combinatorial interpretation to their extreme points. Finally, we show that planar cube-free median graphs can be recognized in linear time.

2. Preliminaries

The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G\Box H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The *r-cube* Q_r is the Cartesian product of r copies of the complete graph on two vertices K_2.

The *interval* $I(u, v)$ between two vertices u and v in G is the set of all vertices on shortest paths between u and v. A subgraph H of G is *convex* if we have $I(u, v) \subseteq V(H)$ for every $u, v \in V(H)$. A graph G is a *median graph* if $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$ holds for every triple of vertices u, v, and w. The vertex of this intersection is called the *median* of the triple u, v, w. It is easy to see that median graphs are bipartite and that the Cartesian product operation preserves median graphs. In addition, a median graph cannot have convex cycles of length greater than 4.

Let $G = (V, E)$ be a graph, V_1 and V_2 subsets of V with nonempty intersection, and $V = V_1 \cup V_2$. Suppose that V_1 and V_2 induce isometric subgraphs of G and that no vertex of $V_1\setminus V_2$ is adjacent to a vertex of $V_2\setminus V_1$. In addition, let $V_1 \cap V_2$ be a convex set in G. Then the *convex expansion* of a graph G with respect to V_1 and V_2 is the graph obtained from G by the following procedure:

(i) replace each vertex $v \in V_1 \cap V_2$ by vertices v_1, v_2 and insert the edge v_1v_2.
(ii) insert edges between v_1 and the neighbors of v in $V_1\setminus V_2$ as well as between v_2 and the neighbors of v in $V_2\setminus V_1$.
(iii) insert the edges v_1u_1 and v_2u_2 whenever $v, u \in V_1 \cap V_2$ are adjacent in G.

We also refer to this as the *convex expansion of G over G_0*, where G_0 is a subgraph of G induced by $V_1 \cap V_2$. Mulder [12,13] proved that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of convex expansions.

In the next proposition we recall two properties of cube-free median graphs that will be needed later. The first one was given in [10, Corollary 3] and the second follows from the main result of [9]. However, to be self-contained as much as possible we give here their short (unified) proofs.

Proposition 2.1. *Let G be a cube-free median graph with n vertices, m edges, s squares, and k classes of the relation Θ. Then*

$$s = m - n + 1 \quad \textit{and} \quad k = -m + 2n - 2.$$

Proof. We prove the claim by induction on the number of expansion steps. The statement is true for K_1. So let G be obtained by an expansion from a cube-free median graph G' over G_0. Then G_0 is a tree. Let n', m', k', and s' be the