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Graphs of polyhedra; polyhedra as graphs

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Abstract

Relations between graph theory and polyhedra are presented in two contexts. In the first, the symbiotic dependence between 3-connected planar graphs and convex polyhedra is described in detail. In the second, a theory of nonconvex polyhedra is based on a graph-theoretic foundation. This approach eliminates the vagueness and inconsistency that pervade much of the literature dealing with polyhedra more general than the convex ones.

Keywords: Polygon; 3-Connected planar graph; Steinitz's theorem; Convex polyhedron; Nonconvex polyhedron; Abstract polyhedron

1. Introduction

1.1. Early history

Polyhedra appeared early in the history (as pyramids, dice, and other signs of civilization) and in geometry (Plato, Euclid, Archimedes). However, all these were individual, particular polyhedra. Euclid's enumeration of the regular polyhedra (Platonic solids) was faulted for not defining the class of polyhedra among which he singles out the regular ones. Euclid's defenders have claimed that the term "polyhedron" was understood by the "man in the street" as denoting convex polyhedra; if so, the ancient Greeks were well ahead of our contemporaries. (The lack of precision is one of the unfortunate "traditions" in the theory of polyhedra; at the present time it continues unabated, on the Web and elsewhere.) No general definition of polyhedra appears till much after Euclid, and even then in strange forms. For example, at the end of the XVII century, Ozanam's famous "Dictionnaire Mathematique" declares [54, p. 119]:

> Le POLYEDRE est un corps terminé par plusieurs Plans rectilignes, & inscriptible dans une Sphere, c'est à dire qu'une Sphere peut être décrite à l'entour' en telle sorte que sa surface touche tous les angles solides du Polyedre ... [The polyhedron is a solid bounded by several straight planes, and inscribable in a sphere, that is, a sphere can be described around in such a way that its surface touches all the solid angles [[vertices]] of the polyhedron.]

Even in the middle of the eighteenth century, Euler discussed his famous theorem $V - E + F = 2$ without specifying what are the polyhedra to which it applies. Apparently he had convex polyhedra in mind, and certainly many of the successors did as well—although they mostly did not define what are the polyhedra about which they are claim to be

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proving theorems. The discovery of polyhedra to which Euler's theorem does not apply led to the pithy comment by Hessel [37]:

> Andere ausgezeichnete Mathematiker (Legendre, Cauchy, Gergonne, Rothe und Steiner) haben Beweise für die allgemeine Gültigkeit des Satzes geliefert. Indessen leidet derselbe Ausnahmen. [Other excellent mathematicians (Legendre, Cauchy, Gergonne, Rothe and Steiner) gave proofs for the general validity of the theorem. But in fact, it suffers from exceptions.]

1.2. More recent developments concerning convex polyhedra

During the 19th century, the center of attention concerning polyhedra switched to (more-or-less explicitly declared) convex ones; we shall later return to the less numerous but still important considerations of other polyhedra. However, a sense of naive trust in the benevolent nature of mathematical objects persisted. Even such otherwise very critical thinkers as Thomas Kirkman seemed to believe that if you draw in the plane a figure that seems to represent a polyhedron, there actually is a convex polyhedron that looks much like this figure (see [40]). The first to publicly question this attitude was Ernst Steinitz, in Section 21 of his 1916 contribution [60] to the great "Encyklopedie der mathematischen Wissenschaften". In the following thirteen sections Steinitz shows how one can formulate the criteria that are necessary and sufficient for the existence of a convex geometric polyhedron that is combinatorially given, and establishes that all such convex realizations are determined up to isomorphism of convex polyhedra. The formulation of the result labeled as "the fundamental theorem of the convex types", as given in [60, p. 77] is:

> Jedes K-polyeder ist als konvexes Polyeder realisierbar. [Every Kpolyhedron can be realized by a convex polyhedron.]

The difficult chore of deciphering what this means is probably responsible for the long-lasting ignorance of this basic theorem about convex polyhedra. The sketch of the proof given in [60] was elaborated by Steinitz in notes for lectures given in the early 1920s; these notes were posthumously published (after editing by H. Rademacher), see [61].

The formulation of Steinitz's criteria is quite cumbersome, starting from very general two-dimensional complexes. In retrospect, after absorbing all his definitions and statements it is rather easy to see that Steinitz had reached, in every respect except terminology, the modern formulation, which we shall discuss in the next section. That formulation is graph-theoretic—but in 1916 there was no graph theory he could use. As far as I can tell, a *graph-theoretic* reformulation of Steinitz's theorem was first published in [30] and, in the formulation which is now standard, in [31]. The first direct graph-theoretic proof was published in [18]. Other proofs and generalizations will be discussed in Sections 2 and 3.

1.3. Nonconvex polyhedra

On the other hand, certain *nonconvex polyhedra* seem to have been first discussed and described in a geometric context by Pacioli [55]. This work contains a number of illustrations of such polyhedra, generally considered to have been drawn by Leonardo da Vinci. Somewhat later, two *regular polyhedra* with *pentagrammatic* faces have been described by Kepler [39]. Although forgotten for almost two centuries, Kepler's work resurfaced after these regular polyhedra, together with two additional ones, were independently discovered by Poinsot [56]. Cauchy [8] soon thereafter proved that there are no other regular polyhedra besides the Platonic one and the ones found by Poinsot.

Later in the nineteenth century, many authors discussed various special kinds of nonconvex polyhedra. A survey of the theory of polyhedra as it existed at the end of that century is the well-known book [7] by Brückner. It presented photographs of a huge number of nonconvex polyhedra. It has been asserted that it gives an exhaustive overview of those polyhedra that have a high degree of symmetry—such as isogonal or isohedral polyhedra (the symmetries of which act transitively on their faces or vertices, respectively). Unfortunately, this book is completely noncritical and is, in fact, internally inconsistent and misleading. I believe that the problems caused by vague definitions (which are, moreover, often ignored by their authors), contributed to the lack of interest in nonconvex polyhedra throughout most of the 20th century.

The central obstacle to any coherent theory of polyhedra more general than the convex ones is the difficulty of defining precisely what objects should be awarded that designation. In Sections 4 and 5 we shall detail the difficulties and the attempts to overcome them. The solution I shall present in Section 7 is based on description of the combinatorial structure of very general abstract polyhedra by certain graphs characterized by their properties, and presenting geometric