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Mobility of vertex-transitive graphs

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Abstract

We define the mobility of a graph automorphism as the minimum distance between a vertex of the graph and its image under the automorphism, and the absolute mobility of a graph as the maximum of the mobilities of its automorphisms. In this paper, we investigate the mobility of certain classes of graphs, in particular, Cartesian and lexicographic products, vertex-transitive graphs, and Cayley graphs.

Keywords: Mobility; Automorphism; Cartesian product; Lexicographic product; Vertex-transitive graph; Cayley graph; Kneser graph; Johnson graph

1. Introduction

In 1979, Nowakowski and Rival [12] characterized reflexive graphs (that is, graphs with a loop at every vertex) with the property that every endomorphism of the graph (that is, homomorphism of the graph into itself) fixes an edge. While they also observed that there are no reflexive graphs with the property that every endomorphism fixes a vertex, in 2001, Nowakowski [11] asked if it might be possible to characterize non-reflexive graphs with this property. Subsequently, Nowakowski and Šajna limited the question to graph automorphisms and defined the following terms.

Definition 1. Let X be a graph and $\alpha \in \mathrm{Aut}(X)$. The *mobility* of the automorphism α, denoted by $\mathrm{mob}(\alpha)$, is defined to be $\min\{\mathrm{d}_X(x, \alpha(x)) : x \in V(X)\}$, where $\mathrm{d}_X(x, y)$ (or simply $\mathrm{d}(x, y)$) denotes the distance between vertices x and y in X. The *absolute mobility* of the graph X, denoted by $\mathrm{am}(X)$, is defined to be $\max\{\mathrm{mob}(\alpha) : \alpha \in \mathrm{Aut}(X)\}$. The *relative mobility* of a connected graph X is defined as $\mathrm{rm}(X) = \mathrm{am}(X)/\mathrm{diam}(X)$, where $\mathrm{diam}(X)$ is the diameter of X.

We remark that all graphs in this paper are assumed to be finite and simple. Unless otherwise specified, we shall use standard notation and terminology from [3] for graphs, and from [4] for permutation groups.

It is easy to see that relative mobility of any connected graph lies in the interval [0, 1]. More precisely, it is bounded above by $\mathrm{rad}(X)/\mathrm{diam}(X)$, where $\mathrm{rad}(X) = \min\{\mathrm{ecc}(v) : v \in V(X)\}$ is the *radius* of X and $\mathrm{ecc}(v) = \max\{\mathrm{d}_X(v, x) : x \in V(X)\}$ is the *eccentricity* of the vertex v. This bound is sharp, that is, there are graphs with absolute mobility equal

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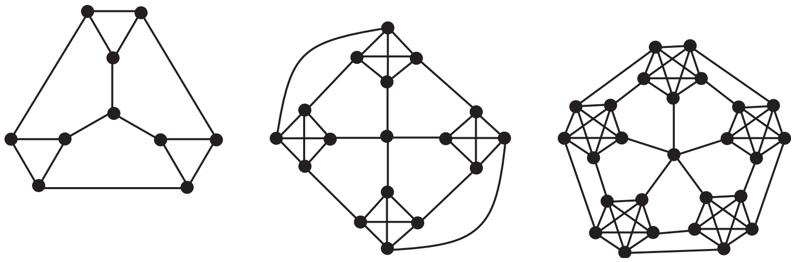

Fig. 1. A family of regular graphs with mobility 0.

to their radius. The following lemma gives a better upper bound for graphs that are connected but not 2-connected. Recall that the *centre* of a connected graph is the set of all vertices x with $\mathrm{ecc}(x) = \mathrm{rad}(X)$, and a *block* is a maximal connected subgraph without a cut-vertex.

Lemma 2. *The centre of a connected graph X is contained in a block B and* $\mathrm{am}(X) \leqslant \mathrm{am}(X[B]) \leqslant \mathrm{rad}(X[B])$. *In particular, if the centre of X comprises a single vertex, then* $\mathrm{am}(X) = 0$.

Proof. The statement of the lemma is trivial for 2-connected graphs, so we may assume that X is not 2-connected. Since every automorphism of a graph fixes its centre setwise, it suffices to show that the centre of a graph lies in a block.

Suppose $x, y \in V(X)$ are two vertices in the centre of X that lie in distinct blocks B_x and B_y, respectively. Since x and y lie in the centre, we have $\mathrm{ecc}(x) = \mathrm{rad}(X) = \mathrm{ecc}(y)$, and since they lie in distinct blocks, there exists a cut-vertex c on an isometric $x - y$ path P that is distinct from both x and y (otherwise, say if $c = x$, then x lies in B_y). We will show that $\mathrm{ecc}(c) < \mathrm{rad}(X)$, contradicting the definition of the radius. Note that every path between a vertex in B_x and a vertex in B_y must contain the cut-vertex c since $X[B_x \cup B_y]$ is not 2-connected.

Take any $w \in V(X), w \neq c$. If $w \in B_x$, then $\mathrm{d}(w, c) < \mathrm{d}(w, y) \leqslant \mathrm{rad}(X)$. Similarly, if $w \in B_y$, then $\mathrm{d}(w, c) < \mathrm{d}(w, x) \leqslant \mathrm{rad}(X)$. If $w \notin B_x \cup B_y$, however, then either an isometric $w - x$ path or an isometric $w - y$ path must pass through c since otherwise c would lie on a cycle. Hence either $\mathrm{d}(w, c) < \mathrm{d}(w, x) \leqslant \mathrm{rad}(X)$ or $\mathrm{d}(w, c) < \mathrm{d}(w, y) \leqslant \mathrm{rad}(X)$. Thus for any vertex $w \in V(X)$, we have $\mathrm{d}(w, c) < \mathrm{rad}(X)$, whence $\mathrm{ecc}(c) < \mathrm{rad}(X)$, a contradiction. We conclude that the centre of X is contained in a block B and therefore $\mathrm{am}(X) \leqslant \mathrm{am}(X[B]) \leqslant \mathrm{rad}(X[B])$. □

Nowakowski and Šajna asked whether it might be possible to characterize graphs with a given relative mobility, especially those with relative mobility 0 or 1. Observe that a graph has relative mobility 0 if and only if each of its automorphisms fixes a vertex, and has relative mobility 1 if and only if it has an automorphism that maps every vertex to a vertex at diameter distance. Clearly, the latter property requires a certain degree of symmetry in the graph. We shall briefly discuss graphs of relative mobility 0 and 1 in the next two sections.

2. Graphs of relative mobility 0

It is easy to find graphs, even regular graphs, with relative mobility 0. Fig. 1 shows the first three members of a family of regular graphs of mobility 0 comprising one graph for every degree $d \geqslant 3$. Note that the centre of each graph in this family consists of a single vertex. We have also found a regular graph with mobility 0 in which no vertex is fixed by all automorphisms (Fig. 2). This example can be easily generalized to yield an infinite family of such graphs. However, as we shall see in Section 5, there are no vertex-transitive graphs of relative mobility 0.

3. Graphs of relative mobility 1

Examples of graphs with relative mobility 1 include complete graphs, complete graphs minus a 1-factor, complete bipartite and complete multipartite graphs, cycles, and circulant graphs. If a graph X has relative mobility 1, then each vertex must be at diameter distance from another vertex in the graph. Hence the eccentricity of each vertex must be the