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Polynomial-time dualization of r-exact hypergraphs with applications in geometry

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ABSTRACT

Let $\mathcal{H} \subseteq 2^V$ be a hypergraph on vertex set V. For a positive integer r, we call $\mathcal{H}$ r-exact if any minimal transversal of $\mathcal{H}$ intersects any hyperedge of $\mathcal{H}$ in at most r vertices. This class includes several interesting examples from geometry, e.g., circular-arc hypergraphs ($r = 2$), hypergraphs defined by sets of axis parallel lines stabbing a given set of α-fat objects ($r = 4\alpha$), and hypergraphs defined by sets of points contained in translates of a given cone in the plane ($r = 2$). For constant r, we give a polynomial-time algorithm for the duality testing problem of a pair of r-exact hypergraphs. This result implies that minimal hitting sets for the above geometric hypergraphs can be generated in output polynomial time.

1. Introduction

Let V be a finite set, and $\mathcal{H} \subseteq 2^V$ be a hypergraph (a family of subsets) on V. A set $X \subseteq V$ is called a *transversal* (or a *hitting set*) if $X \cap H \neq \emptyset$ for all $H \in \mathcal{H}$, and is called further a *minimal* transversal if any proper subset of X is not a transversal for $\mathcal{H}$.

We assume throughout that $\mathcal{H}$ is *Sperner*, i.e., no hyperedge of $\mathcal{H}$ contains another. If this is not the case then we can safely ignore the inclusionwise larger hyperedges since any minimal transversal of the resulting hypergraph is also a minimal transversal of the original hypergraph.

Let $\mathcal{H}^d \subseteq 2^V$ be the set of all minimal transversals for $\mathcal{H}$ (also called the *dual* hypergraph of $\mathcal{H}$). Finding all minimal transversals for a given hypergraph $\mathcal{H}$, i.e., generating $\mathcal{H}^d$, is a well-known problem called the *hypergraph transversal problem* [2], which has received considerable attention in the literature (see, e.g., [3,11,13,22,33,37]), since it is known to be polynomially or quasi-polynomially equivalent to many problems in various areas, such as artificial intelligence (e.g., [11,28]), database theory (e.g., [35]), distributed systems (e.g., [19,26]), machine learning and data mining (e.g., [1,8,23]), mathematical programming (e.g., [6,29]), matroid theory (e.g., [31]), and reliability theory (e.g., [10,38]). In this paper, we give a number of further applications in geometry.

Clearly, the size of $\mathcal{H}^d$ can be exponentially larger than $|\mathcal{H}|$, and thus it is natural to look for algorithms whose running time is polynomial in $|\mathcal{H}^d|$. Such an algorithm is said to run in incremental polynomial time if for any k, the time required to find k minimal transversals is polynomial in $|V|$, $|\mathcal{H}|$, and k.

The currently fastest known algorithm [18] for solving the hypergraph transversal problem runs in quasi-polynomial time $|V| \cdot N^{o(\log N)}$, where N is the combined input and output size: $N = |\mathcal{H}| + |\mathcal{H}^d|$. A number of quasi-polynomial-time algorithms with some other desirable properties also exist [41,15,21,9]. While it is still open whether the problem can be solved in polynomial time for arbitrary hypergraphs, polynomial-time algorithms exist for several classes of hypergraphs, e.g. hypergraphs of bounded edge size [4,11], of bounded degree [36,12], of bounded edge intersections [5], of bounded conformality [5], of bounded treewidth [12], of bounded latency [34] and read-once (exact) hypergraphs [42].

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Almost all previously known polynomial-time algorithms for the hypergraph transversal problem assume that at least one of the hypergraphs $\mathcal{H}$ or $\mathcal{H}^d$ (i) has bounded size $\min\{|\mathcal{H}|, |\mathcal{H}^d|\} \le k$, or (ii) is k-conformal,[1] or (iii) is k-degenerate,[2] for a constant k. One can verify that, except for [34], all the special classes mentioned above belong to one of these categories.

In this paper, we shall extend these polynomially dualizable classes of hypergraphs as follows. Given an integer $r \ge 1$, let us say that a hypergraph $\mathcal{H} \subseteq 2^V$ is *r-exact* if

$$|H \cap T| \le r, \quad \text{for all } H \in \mathcal{H} \text{ and } T \in \mathcal{H}^d. \tag{1}$$

As we shall see later, these hypergraphs can be recognized in polynomial time. Note that this class of hypergraphs includes the case when $\max\{|H| : H \in \mathcal{H}\} \le r$ or $\max\{|T| : T \in \mathcal{H}^d\} \le r$.

When $\mathcal{H}$ satisfies (1) with $r = 1$, $\mathcal{H}$ is called an *exact* or *read-once* hypergraph, and the problem of finding $\mathcal{H}^d$ is known to be solvable with polynomial delay, using a simple backtracking approach [42]. However, this technique does not seem to generalize for $r \ge 2$. Using a more sophisticated technique, we show in Section 4 that the problem can still be solved in incremental polynomial time, for any hypergraph satisfying (1). The best previously known result for this class was quasi-polynomial: $\mathrm{poly}(n, |\mathcal{H}^d|)|\mathcal{H}|^{O(\log|\mathcal{H}|)}$ [30] (which gives in fact a global parallel algorithm running in polylogarithmic time and requiring a quasi-polynomial number of processors).

Theorem 1. *Let $\mathcal{H}$ be an r-exact hypergraph with m edges and n vertices, and k be a given positive integer. Then for $r = O(1)$, we can find k minimal transversals of $\mathcal{H}$ in time* $\mathrm{poly}(n, m, k)$.

As consequences of Theorem 1, we obtain incremental polynomial-time algorithms for finding:

- all minimal hitting sets, and all minimal set covers, for a circular-arc hypergraph (see, e.g., [17])—this generalizes known results for interval hypergraphs [12,5];
- all minimal hitting sets, and all minimal set covers, for a hypergraph defined by a set of points and given translates of a certain cone in the plane;
- all minimal subsets of a given set of axis parallel hyperplanes, hitting a set of comparable fat objects in $\mathbb{R}^d$, for fixed d (see, e.g., [20,32] for the corresponding optimization problems).

The enumeration of minimal geometric hitting sets, such as the ones described above, may arise in various areas such as computational geometry, machine learning, and data mining [14]. Moreover, efficient enumeration algorithms are also known to be useful in developing exact algorithms, fixed-parameter tractable algorithms, and polynomial-time approximation schemes for the corresponding optimization problems (see, e.g., [39,16,25]).

The rest of the paper is organized as follows. In Section 2 we recall that the class of r-exact hypergraphs can be recognized in polynomial time,[3] and that, unlike for the case $r = 1$, it does not generalize read-r Boolean functions. In Section 3, we give the details of the geometric applications listed above. Finally, we prove Theorem 1 in Section 4.

2. Preliminaries

2.1. Notation

Let V be a finite set of size $|V| = n$. For a hypergraph $\mathcal{H} \subseteq 2^V$ and a subset $S \subseteq V$, we use the following notation: $\mathcal{H}_S$ denotes the sub-hypergraph induced by the vertices in S, i.e., $\mathcal{H}_S = \{H \in \mathcal{H} \mid H \subseteq S\}$, and $\mathcal{H}^S$ denotes the projection of $\mathcal{H}$ on S, i.e., $\mathcal{H}^S = \mathrm{minimal}(\{H \cap S \mid H \in \mathcal{H}\})$, where for any hypergraph $\mathcal{H}'$, $\mathrm{minimal}(\mathcal{H}')$ denotes the Sperner hypergraph that we get by keeping the inclusionwise minimal set of hyperedges from $\mathcal{H}'$. Note that if $\mathcal{H}$ satisfies (1) then so do the hypergraphs $\mathcal{H}_S$ and $\mathcal{H}^S$, for any $S \subseteq V$.

For a hypergraph $\mathcal{H}$, its *transpose* is the hypergraph $\mathcal{H}'$ obtained by switching the roles of vertices and edges in $\mathcal{H}$: $\mathcal{H}' = \{\{H \in \mathcal{H} : H \ni v\} : v \in V\}$.

2.2. Recognizing r-exact hypergraphs

Given a subset $S \subseteq V$ of vertices, [7] gave a criterion for deciding whether S is a *sub-transversal* of $\mathcal{H}$, i.e., there is a minimal transversal $T \in \mathcal{H}^d$ such that $T \supseteq S$. In general, testing whether S is a sub-transversal is an NP-hard problem even if $\mathcal{H}$ is a graph (see [4]). However, if $|S|$ is bounded by a constant, then such a check can be done in polynomial time. This observation was used to solve the hypergraph transversal problem in polynomial time for hypergraphs of bounded edge size in [4], or more general of bounded conformality [5]. We can also use it to recognize r-exact hypergraphs in polynomial time.

To describe this criterion, we need a few more definitions. For a subset $S \subseteq V$, and a vertex $v \in S$, let $\mathcal{H}_v(S) = \{H \in \mathcal{H} \mid H \cap S = \{v\}\}$. A selection of $|S|$ hyperedges $\{H_v \in \mathcal{H}_v(S) \mid v \in S\}$ is called *covering* if there exists a hyperedge $H \in \mathcal{H}_{V \setminus S}$ such that $H \subseteq \bigcup_{v \in S} H_v$.

1 A hypergraph is said to be k-conformal [2] if any set $X \subseteq V$ is contained in a hyperedge of $\mathcal{H}$ whenever each subset of X of cardinality at most k is contained in a hyperedge of $\mathcal{H}$.

2 A hypergraph $\mathcal{H}$ is said to be k-degenerate [12] if for every set $X \subseteq V$, the minimum degree of a vertex in the induced hypergraph $\mathcal{H}_X$ on X is at most k.

3 This was also observed in [30]; we include the proof for completeness.