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Clonal sets in $GF(q)$-representable matroids

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ABSTRACT

Whittle [12] conjectured that if M is a 3-connected quaternary matroid with a clonal pair $\{e, f\}$, then $M \setminus e, f$ and $M/e, f$ are both binary. In this paper we show that for $q \in \{4, 5, 7, 8, 9\}$ if M is a 3-connected $GF(q)$-representable matroid with a clonal set X of size $q-2$, then $M \setminus X$ and M/X are binary.

1. Introduction

Our notation and terminology will generally follow Oxley [6] with one exception: we use $\mathrm{si}(M)$ and $\mathrm{co}(M)$ to denote the simplification and cosimplification of a matroid M, respectively.

One of the most interesting research problems in matroid theory is to find a way to determine whether a given matroid is representable over a fixed finite field. Tutte [10] proved a matroid is binary if and only if it does not contain a minor isomorphic to $U_{2,4}$. More generally, Rota [8] conjectured that for any finite field F, there are only finitely many minor-minimal non-F-representable matroids. Rota's conjecture has been shown to be true for all fields F with $|F| \le 4$. The next result is due to Geelen, Gerards, and Kapoor [3].

Theorem 1.1. *A matroid M is $GF(4)$-representable if and only if it does not contain a minor isomorphic to any of $U_{2,6}$, $U_{4,6}$, F_7^-, $(F_7^-)^*$, P_6, P_8, and P_8''.*

Two elements are *clones* if the map that interchanges the two elements and fixes all other elements is an automorphism of the matroid. Clones have recently become an important subject in the study of the representability of matroids over finite fields [2,4]. It is clear that for any field F, the class of F-representable matroids is closed under direct sums and 2-sums. Therefore, we may focus only on 3-connected matroids when we study the F-representability. In the case that M has a clonal pair, the next result [2] provides a simpler alternative to Theorem 1.1.

Theorem 1.2. *Let M be a 3-connected matroid with a clonal pair X. If M is not $GF(4)$-representable, then M has a minor using X that is isomorphic to one of $U_{2,6}$, $U_{4,6}$, and P_6.*

Theorem 1.2 characterizes 3-connected non-$GF(4)$-representable matroids with a clonal pair. For 3-connected $GF(4)$-representable matroids, Whittle [12] conjectured the following.

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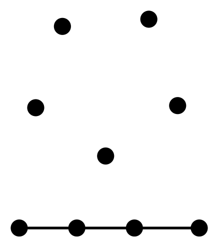

Fig. 1. The matroid $\bar{U}_{3,7}$.

Conjecture 1.3. *If M is a 3-connected $GF(4)$-representable matroid with a clonal pair X, then $M \setminus X$ is binary.*

Conjecture 1.3, if true, would reduce the problem of proving some particular matroid is non-$GF(4)$-representable to a problem of proving a related matroid is not binary.

Theorem 1.4. *Let M be a 3-connected matroid with a clonal pair $\{e, f\}$. Then*

(1) *If $M \setminus e, f$ is non-binary, then M has a $U_{2,6}$-minor using $\{e, f\}$.*
(2) *If $M/e, f$ is non-binary, then M has a $U_{4,6}$-minor using $\{e, f\}$.*

Since the matroid $U_{2,6}$ is not $GF(4)$-representable, Whittle's conjecture is an immediate consequence of Theorem 1.4. We now study $GF(q)$-representable matroids that have a clonal set of size $q-2$. We conjecture the following.

Conjecture 1.5. *Let M be a 3-connected $GF(q)$-representable matroid with a clonal set of size $q-2$. Then $M \setminus X$ and M/X are both binary.*

Let $\bar{U}_{r,k}$ be the matroid obtained from the uniform matroid $U_{r,q}$ by freely adding two points on a line. Fig. 1 shows a geometric representation of the matroid $\bar{U}_{3,7}$.

Theorem 1.6. *Conjecture 1.5 holds if and only if each of the matroids $\bar{U}_{r,q}$, $2 \le r \le \lfloor q/2 \rfloor$ is not $GF(q)$-representable.*

Now to solve Conjecture 1.5, we need to study the representability of the matroids $\bar{U}_{r,q}$, $2 \le r \le \lfloor q/2 \rfloor$. This problem is probably in the same difficulty level as the problem on studying the representability of uniform matroids. Nevertheless, we can solve some of the cases when the rank is small. As a consequence, we obtain the following result.

Theorem 1.7. *For $q \in \{4, 5, 7, 8, 9\}$, if M is a 3-connected $GF(q)$-representable matroid with a clonal set X of size $q-2$, then both $M \setminus X$ and M/X are binary.*

The paper is organized as follows: Section 2 contains some preliminary lemmas on connectivity and clones that will be use in later sections; we present the proof of Theorem 1.4 in Section 3; the proofs of Theorems 1.6 and 1.7 are presented in Section 4.

2. Preliminaries

Let $M = (E, r)$ be a matroid where r is the rank function. The connectivity function of M, denoted by λ_M, is defined as $\lambda_M(A) = r(A) + r(E - A) - r(M)$. Then A is k-separating if and only if $\lambda_M(A) \le k - 1$. It is easily verified that $\lambda_M(A) = r(A) + r^*(A) - |A|$. The *coclosure* of a set $X \subseteq E(M)$, denoted by $\text{cl}^*(X)$, is the closure of X in M^*. We omit the proof of the next lemma which is straightforward.

Lemma 2.1. *Let M be a matroid and let $(A, \{x\}, B)$ be a partition of $E(M)$. Then $x \in \text{cl}^*(A)$ if and only if $x \notin \text{cl}(B)$.*

Let (A, B) be a k-separation of the matroid M. We call (A, B) a *minimal k-separation* if either $|A| = k$ or $|B| = k$. We call (A, B) an *exact k-separation* or A is *exactly k-separating* if $\lambda_M(A) = k - 1$. An element $x \in E(M)$ is in the *guts* of (A, B) if x belongs to the closure of both A and B. Dually, x is in the *coguts* of (A, B) if x belongs to the coclosure of both A and B. The next lemma follows easily from the definitions.

Lemma 2.2. *Let (A, B) be an exact k-separation of matroid M and let $x \in B$. Then*

- *$A \cup \{x\}$ is exactly k-separating if x belongs to either the guts or the coguts of (A, B), but not both.*
- *$A \cup \{x\}$ is exactly $(k-1)$-separating if x belongs to both the guts and the coguts of (A, B).*
- *$A \cup \{x\}$ is exactly $(k+1)$-separating if x belongs to neither the guts nor the coguts of (A, B).*

Suppose x is an element of the matroid M and let (A, B) be a k-separation of $M \setminus x$. Then x *blocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. Now let (A, B) be a k-separation of M/x. Then x *coblocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. The following lemma also follows easily from definitions.