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Note

Tessellations with arbitrary growth rates

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ABSTRACT

A *tessellation* is taken to be an infinite, 1-ended, 3-connected, locally finite, and locally cofinite plane map. When such a tessellation is the induced graph of a tiling of the hyperbolic plane, it is known that the asymptotic growth of the tessellation is exponential. We address an unpublished conjecture of Watkins, that growth rates of such tessellations can be made arbitrarily close to 1. Given any real number $\xi > 1$, we use analytic methods to construct a tessellation with growth rate ξ.

1. Introduction

A *tessellation* is an infinite, one-ended plane map which is also 3-connected, locally finite, and locally cofinite; that is, each face has finite covalence. The growth of a tessellation is measured from a central face or vertex, called its *root*; if F_n is the set of faces in the nth corona of a Bilinski diagram for a tessellation T, then the *growth rate of* T is defined to be $\gamma(T) = \frac{1}{R}$, where $0 < R < \infty$ is the radius of convergence of the power series

$$\phi(z) = \sum_{i=0}^{\infty} |F_i| z^i.$$

This extends the definition of growth used by Moran [4], dealing with face-homogeneous tessellations, and by Graves et al. [2], dealing with edge-homogeneous tessellations.

The *edge-symbol* of an edge in a tessellation is the 4-tuple $\langle p, q; r, s\rangle$ of integers at least 3 where p and q are the valences of vertices incident with the edge and r and s are the covalences of faces incident with the edge. The tessellation is *edge-homogeneous* if every edge has the same edge-symbol. In [3], Grünbaum and Shephard proved that the edge-symbol of a tessellation uniquely identifies the tessellation up to isomorphism, and that edge-homogeneous tessellations are edge-transitive. With uniqueness established, Graves et al. [2] determined that the growth rate of an edge-homogeneous tessellation is dependent solely upon a function of the parameters of the edge-symbol; their result is stated here.

Proposition 1 (*[2], Theorem 4.1*)**.** *Let the function* $g : \{t \in \mathbb{Z} : t \geq 4\} \longrightarrow [1, \infty)$ *be given by*

$$g(t) = \frac{1}{2}\left(t - 2 + \sqrt{(t-2)^2 - 4}\right).$$

Let T *be an edge-homogeneous tessellation with edge symbol* $\langle p, q; r, s\rangle$*, and let*

$$t = \left(\frac{p+q}{2} - 2\right)\left(\frac{r+s}{2} - 2\right).$$

Then exactly one of the following holds:

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1. *the growth rate of T is $\gamma(T) = g(t)$;*
2. *the edge-symbol of T or its planar dual is $\langle p, 3; 4, 4\rangle$ with $p \geq 6$, and the growth rate of T is $\gamma(T) = g(t-1)$.*

Since this g is an increasing function in t and $t \geq 5$ when T is a tessellation of the hyperbolic plane, the least rate of exponential growth for an edge-homogeneous tessellation is $g(5) = \frac{1}{2}(3+\sqrt{5})$, as tessellations of the Euclidean plane have quadratic growth. This led to an unpublished conjecture by Mark Watkins (developed in the author's dissertation [1]) that relaxing the hypotheses to face-homogeneous tessellations would give a least rate of growth closer to 1. Watkins also conjectured that total removal of the constraints of homogeneity and symmetry would allow the construction of a tessellation T with growth rate $\gamma(T)$ satisfying $1 < \gamma(T) < 1 + \varepsilon$ for arbitrarily $\varepsilon > 0$. The main result of this paper is a stronger result, providing a construction for a tessellation T with $\gamma(T) = \xi$ for any $\xi > 1$.

2. Preliminaries

Definition 2. Let T be a tessellation of the plane. A *Bilinski diagram* for T is a labeling of T which proceeds inductively as follows:

- Let T_0 be a submap of T induced by a single vertex or a single face of T.
- Let U_0 be the vertex set of T_0 and F_0 the face set of T_0.
- For all $n \geq 1$, let F_n be the set of faces incident with a vertex of U_{n-1} not included in F_{n-1}.
- For all $n \geq 1$, let U_n be the set of vertices incident with faces in F_n not included in U_{n-1}.

The induced subgraph $\langle F_n\rangle$ is the nth *corona*.

The structure of a Bilinski diagram makes it more convenient to consider distance in terms of regional or coronal distance, rather than minimal path length. Along these lines, Moran [4] defined the growth rate of a tessellation to be

$$\lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} \tag{1}$$

provided the limit exists, where

$$\tau(n) = \sum_{k=0}^{n} |F_k| .$$

While this is a useful definition, the question of the existence of the limit is vexing, as it leads to a necessary discussion of balanced versus unbalanced growth. This is avoided by instead making the following definition.

Definition 3. Let T be a tessellation labeled as a Bilinski diagram; then the power series

$$\phi(z) = \sum_{n=0}^{\infty} |F_n| z^n \tag{2}$$

converges within some radius R about 0. When $0 < R < \infty$, we define the *growth rate of* T to be $\gamma(T) = 1/R$.

Notice that

$$\frac{\phi(z)}{1-z} = \sum_{n=0}^{\infty} \tau(n) z^n.$$

If the radius of convergence R of $\phi(z)$ is such that $0 < R < 1$, we get

$$\lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} = \limsup_{n\to\infty} \left|\frac{\tau(n+1)}{\tau(n)}\right| = \frac{1}{R} \tag{3}$$

by the ratio test for convergence of power series, when the left-hand limit exists.

Proposition 4. *Let T_1 and T_2 be tessellations, and let $\left|F_{i,n}\right|$ be the number of faces in the nth corona of a Bilinski diagram of T_i for $i = 1, 2$. Suppose that $\left|F_{1,n}\right| \leq \left|F_{2,n}\right|$ for all but finitely many n. Then $\gamma(T_1) \leq \gamma(T_2)$.*

Proof. Let $\phi_1(z) = \sum \left|F_{1,n}\right| z^n$ and $\phi_2(z) = \sum \left|F_{2,n}\right| z^n$, and let R_i be the radius of convergence of $\phi_i(z)$ about 0. Then since $\left|F_{1,n}\right| \leq \left|F_{2,n}\right|$ for sufficiently large n, and

$$\gamma(T_1) = \frac{1}{R_1} = \limsup_{n\to\infty} \sqrt[n]{|F_1, n|} \leq \limsup_{n\to\infty} \sqrt[n]{|F_2, n|} = \frac{1}{R_2} = \gamma(T_2)$$

for $i = 1, 2$, we have $\gamma(T_1) \leq \gamma(T_2)$ as desired. $\square$