Discrete Mathematics 310 (2010) 255–259

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cycle-regular graphs of $(0, \lambda)$-graph type

Nawel Kahoul*, Abdelhafid Berrachedi

Département de recherche opérationnelle Faculté de Mathématiques, USTHB BP 32 El Alia, Bab-Ezzouar 16111, Alger, Algérie

ARTICLE INFO

Article history:
Received 10 July 2007
Accepted 11 September 2008
Available online 19 October 2008

Keywords:
Hypercubes
Odd graphs
Semi regular graphs
$(0, \lambda)$-graphs
Cycle regular graphs

ABSTRACT

In this paper we study $[3, 1, 6]$-*cycle-regular graphs*, a subclass of the *cycle-regular graphs* introduced by M. Mollard. These graphs are a generalization of $(0, \lambda)$-*graphs* introduced by H.M. Mulder. Amongst other we obtain a characterization of the subgraph induced by the two middle levels of a hypercube.

1. Introduction

Unless specified otherwise, all graphs in this paper are finite, simple, undirected and connected. A graph G [4] will have a vertex set $V(G)$ and an edge set $E(G)$. In the sequel, we always write V (resp. E) instead of $V(G)$ (resp. $E(G)$), except in the case where two or more graphs are considered. Thus, we simply write $G = (V, E)$. The order of G is the number of its vertices. The graph on n pairwise adjacent vertices is denoted by K_n. The *neighbourhood* of a vertex $u \in V$ will be denoted by $N(u)$. The *degree* of a vertex u of G and the *minimum degree* of vertices of G will be denoted by $d(u)$ and $\delta(G)$, respectively. A bipartite graph is *semiregular* if the vertices in the same part of the bipartition have the same degree.

In this paper, an *elementary path* $P_{\mu+1}$ (also called a (u_0, u_μ)-*path*) of length μ (in G) is a sequence $u_0, \ldots, u_\mu$ of pairwise distinct vertices except possibly u_0 and u_μ, where $u_i u_{i+1} \in E$ for $i = 0, \ldots, \mu - 1$. An *elementary cycle* of length μ (in G) is a (u_0, u_μ)-path with $u_0 = u_\mu$ and is called a μ-*cycle*. Both a (u_0, u_μ)-path and a μ-cycle are *induced* if any *two* non-consecutive vertices are not adjacent. The *girth* of a graph G is the length of the shortest cycle in G. A (u_0, u_μ)-path belongs to an elementary cycle $v_0, \ldots, v_{\nu-1}, v_0$ if $\mu \le \nu$ and $u_i = v_i$ for some $0 \le i \le \nu - 1$.

The *distance* between two vertices u and v in G is the length of the shortest (u, v)-path and is denoted by $d(u, v)$. The *diameter* of the graph G is $\operatorname{diam}(G) = \max\{d(u, v) : u, v \in V\}$. For any vertex $u \in V$, we denote by $N_i(u) = \{v \in V : d(u, v) = i\}$. For a given $u \in V$ and a positive integer n such that $n = \max_{v \in V} d(u, v)$, the partition of V into $\{N_i(u) : i = 0, \ldots, n\}$ is a *level decomposition of G from u*. The set $N_i(u)$ is called the ith *level*. In this paper we are mostly interested in some specific level decomposition, where the vertex in the bottom level is not of interest. Then we will write N_i for the ith *level*. In such a decomposition, edges connect vertices in consecutive levels or in the same level. Given $u \in V$ and a level decomposition $\{N_i : i = 0, \ldots, n\}$ from u, we define for $v \in N_i$ the number $d^-(v) = |N(v) \cap N_{i-1}|$ (resp. $d^+(v) = |N(v) \cap N_{i+1}|$).

The *Categorical product* $G \times H$ of two graphs G and H has a vertex-set $V(G \times H) = V(G) \times V(H)$ and two vertices (u, v), (u', v') in $G \times H$ are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

The *hypercube* Q_n has $V = \{A : A \subseteq \{1, 2, \ldots, n\}\}$ as a vertex-set and two vertices A and B are adjacent if and only if $|A \Delta B| = |(A \setminus B) \cup (B \setminus A)| = 1$. Q_n is regular of degree n and has diameter n (Fig. 1).

* Corresponding author.
E-mail addresses: kahoul.nawel@caramail.com (N. Kahoul), aberrachedi@usthb.dz (A. Berrachedi).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.09.022

The subgraph of Q_n induced by two consecutive levels N_{k-1} and N_k and denoted by L_n^k is semiregular of degrees $n-k+1$ and k, and it has order $\binom{n}{k} + \binom{n}{k-1}$. In particular, the subgraph induced by the two middle levels N_{k-1} and N_k of Q_{2k-1} and denoted by L_{2k-1}^k or more frequently H_k [1–3,5] is regular of degree k. For $k = 3$, we obtain the Desargues graph H_3 (Fig. 2).

The Odd graph O_n has the set $\{A : A \subseteq \{1, 2, \ldots, 2n-1\}; |A| = n-1\}$ as a vertex-set and two vertices are adjacent if their corresponding subsets are disjoint. The odd graph O_n is regular of degree n and is of girth six for $n \geq 4$ (Fig. 3).

2. Preliminary results about the cycle-regular graphs

Mulder [10,11] introduced $(0, \lambda)$-graphs for which each two distinct vertices have either 0 or λ common neighbours. Furthermore, he proved that maximum $(0, \lambda)$-graphs are hypercubes. One way of generalizing this concept is to consider cycle-regular graphs which have some regularity properties and whose maximum graph for particular cases is related to hypercubes.

Definition 1 (*Mollard [9]*). A graph $G = (V, E)$ of girth at least μ ($\mu \geq 2$) is a $[\mu, \lambda]$-cycle-regular graph ($\lambda \geq 1$) if there is a non-empty subset C of elementary cycles in G such that every path $P_{\mu+1}$ in G belongs to exactly λ cycles in C. In the particular case when C is the set of elementary cycles of a given length η ($\eta \geq 2\mu$), we say that G is a *$[\mu, \lambda, \eta]$-cycle-regular graph* (also called a cycle-regular graph).

We can say now that the $(0, \lambda)$-graphs are the *$[2, \lambda-1, 4]$-cycle-regular graphs*. Our study focuses on the *[3, 1, 6]-cycle-regular graphs*, like the odd graphs O_n ($n > 2$). The [3, 1, 6]-cycle-regular graphs are triangle-free, since the triangle cannot belong to an elementary cycle of length greater than three. So for any vertex u, the set $N(u)$ is stable. In this class, we can also consider the subgraph L_n^k.

Mulder [10] and Laborde and Rao Hebbare [6] showed separately that for a given degree and among all the [2, 1, 4]-cycle-regular graphs, the hypercube is of maximum order. On the other hand, Mollard [7] showed that for a given degree, the hypercube is of maximum diameter among these graphs. Furthermore for a given degree n, he showed that H_n is of maximum order among the [3, 1, 6]-cycle-regular graphs [9]. In this paper, we give some new characterizations of H_n in the class of graphs which are [3, 1, 6]-cycle-regular graphs. Moreover, we give other properties of [3, 1, 6]-cycle-regular graphs.

Proposition 1 (*Mollard [9]*). *If $G = (V, E)$ is a $[\mu, \lambda]$-cycle-regular graph of minimal degree $\delta(G) \geq 3$, then G is regular or semi-regular.*

3. [3, 1, 6]-Cycle-Regular Graphs

Mollard [9] gave an upper bound for the order of a [3, 1, 6]-cycle-regular graph of a given degree. Moreover, he gave a characterization of the subgraph H_n.

Proposition 2 (*Mollard [9]*). *Let $G = (V, E)$ be a $[3, 1, 6]$-cycle-regular graph of maximum degree n. Then*

(1) $|V| \leq \binom{2n}{n}$,
(2) $|V| = \binom{2n}{n}$ *if and only if G is H_n.*

To establish the proof of Proposition 2, Mollard used Propositions 3 and 4.

Proposition 3 (*Mollard [9]*). *Let $G = (V, E)$ be a $[3, 1, 6]$-cycle-regular graph and for an arbitrary level decomposition $\{N_0, N_1, \ldots, N_p\}$ of G, let $u \in N_i$. Then $d^-(u) \geq \lceil \frac{i}{2} \rceil$.*

By Proposition 3, Mollard [9] deduced that the diameter of a [3, 1, 6]-cycle-regular graph is at most $2n-1$ for a given maximum degree n.

Proposition 4 (*Mollard [9]*). *Let $G = (V, E)$ be a $[3, 1, 6]$-cycle-regular graph of maximum degree n and $\{N_0, N_1, \ldots, N_p\}$ be a level decomposition from a vertex of degree n. Then for $k = 0, \ldots, n-2$,*

$$|N_{2k+1}| \leq \frac{n}{k+1}\left(\binom{n-1}{k}\right)^2 \quad \text{and} \quad |N_{2k+2}| \leq \frac{n(n-k-1)}{(k+1)^2}\left(\binom{n-1}{k}\right)^2.$$

For a given maximum degree, we show that H_n is of maximum diameter among the [3, 1, 6]-cycle-regular graphs.

Theorem 1. *Let $G = (V, E)$ be a $[3, 1, 6]$-cycle-regular graph of maximum degree $n \geq 2$. Then*

(1) $\mathrm{diam}(G) \leq 2n-1$,
(2) $\mathrm{diam}(G) = 2n-1$ *if and only if G is H_n.*