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The congruence classes of paths and cycles

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ABSTRACT

By traversing square lattices, the cardinality of the set of congruence classes induced by the graph endomorphisms of undirected paths is determined. Enhancing this idea, formulas for the cardinality of the set of endomorphisms and the set of congruence classes of undirected cycles are developed.

1. Introduction and basic results

Counting graph endomorphisms of special classes of graphs has become an often discussed topic in both graph theory and discrete combinatorics recently. Unfortunately, as of now there are almost no published results available on this subject. In [1], S. Arworn describes a method of computing the cardinality of the endomorphism monoids of undirected paths of arbitrary length by using a general square lattice, but as the number of endomorphisms grows very fast with increasing length of the paths, the given formulas get very bulky and inelegant. We picked up the idea and attended to investigate the cardinality of the set of congruence classes induced by the endomorphisms first. Then we count the possible embeddings of the resulting factor graphs into the original graph. This approach leads to structures which can possibly be enhanced to characterize the properties of the endomorphism monoid of other classes of graphs.

Definition 1.1. An undirected graph $G = (V, E)$ with the vertex set $V := \{1, \ldots, n\} \subseteq \mathbb{N}, n \in \mathbb{N}, n \geq 2$ and the edge set defined by $\{x, y\} \in E \Leftrightarrow |x - y| = 1$ is called a **path** with n vertices and denoted by P_n. A graph $G = (V, E')$ with V as above and $E' = E \cup \{1, n\}$ is called a **cycle** with n vertices and denoted by C_n.

Obviously, endomorphisms of a path can be found by picking a fixed starting vertex and moving to vertices which are adjacent to this vertex, as $\varphi \in \mathrm{End}(P_n) \Leftrightarrow \forall x \in \{1, \ldots, n-1\} : \{\varphi(x), \varphi(x+1)\} \in E(P_n)$. We will construct diagrams which describe all possible moves through the edge structure of a path.

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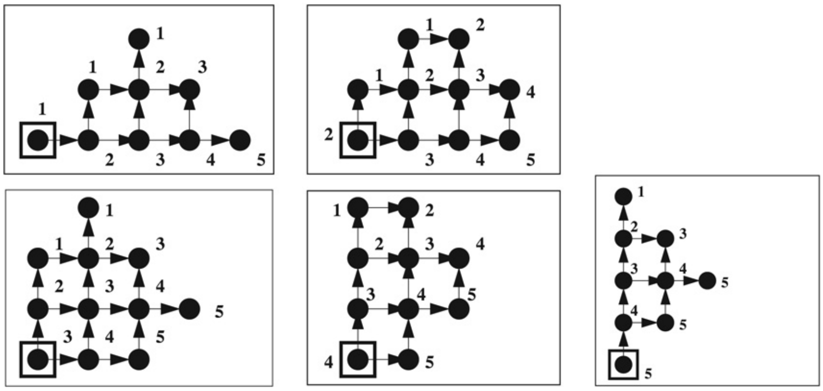
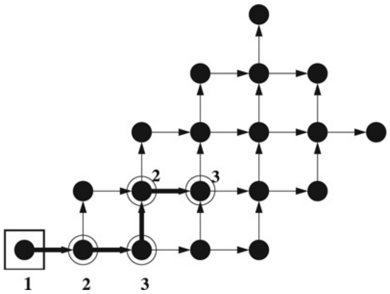


The above diagrams are constructed from a path P_5 by starting at a fixed vertex x (the one in the rectangle). If the starting vertex is 1, we can only move to vertex 2. If the starting vertex is 5, the only vertex which can be reached by an edge is vertex 4. All other vertices are adjacent to two different vertices, thus there are two options in this case. We stop after traversing a path of length $n-1$. Due to the fact that not every vertex has two neighbors, different shapes arise for every starting point as movement is limited when reaching the first or the last vertex.

Example 1.2. An endomorphism of P_5 can be constructed as follows:



Starting at vertex 1 in the bottom row and moving right, right, up, right delivers: 1, 2, 3, 2, 3. The map $\varphi(1) = 1, \varphi(2) = 2, \varphi(3) = 3, \varphi(4) = 2, \varphi(5) = 3$ obviously is an endomorphism of P_5. The congruence classes induced by φ are 1/24/35. Starting at vertex 2 gives, moving in the same shape: 2, 3, 4, 3, 4 – also an endomorphism of P_5, the congruence classes again are 1/24/35.

Definition 1.3. The graph with the vertex set $V = \{v_{i,j} \mid i, j \in \mathbb{N}\}$ and the edge set $E = \{\{v_{i,j}, v_{i+1,j}\}, \{v_{i,j}, v_{i,j+1}\} \mid i, j \in \mathbb{N}\}$ is called a square lattice. A square lattice EG_n with vertex numbers $\Omega(v_{i,j}) := ((i+j-1) \bmod n) + 1$ assigned to the vertex in row i and column j is called a **2-dimensional endogrid**.

Definition 1.4. An n-tuple

$$f = (s, w_1, \ldots, w_{n-1}), \quad s \in \{1, \ldots, n\},\ w_i \in \mathbb{F}_2, i \in \{1, \ldots, n-1\}$$

is called a **rooted binary path** of length n. The number s is called a **root** of f, the $(n-1)$-tuple $(w_1, \ldots, , w_{n-1})$ is called a **binary path** of length $n-1$. The **trail** $tr(f)$ of a rooted binary path f of length n, is defined as $tr(f) := (t_1, \ldots, t_n)$ with $t_1 := s$ and $t_i := s + \sum_{j=1}^{i-1}(-1)^{w_j}$, $(2 \le i \le n)$. A rooted binary path f of length n is called **path-valid** if for $tr(f) = (t_1, \ldots, t_n)$:

$$1 \le t_i \le n, \quad 1 \le i \le n.$$

The set of all path-valid rooted binary paths is called P-Root(n).

The trail function "decodes" the rooted binary path by taking the root and interpreting every 0 as "move right" or "+1" and every 1 as "move up" or "−1".

Theorem 1.5. *The trail $tr(f) = (t_1, \ldots, t_n)$ of a path-valid rooted binary path f of length n is an endomorphism φ of the path P_n if we set $\varphi(i) := t_i$. It is also called the* ***resulting endomorphism*** *of f. Moreover, tr:* P-Root(n) $\to$ End(P_n) *is bijective.*

Proof. We first show that $tr(f)$ for $f \in$ P-Root(n) is an endomorphism of P_n: This follows immediately as the trail of a path-valid rooted binary path of length n consecutively gives the numbers of the image points $\{1, \ldots, n\}$. Now we show that $tr(f)$ is bijective: Suppose that $f = (s, w_1, \ldots, w_{n-1}), g = (t, x_1, \ldots, x_{n-1}) \in$ P-Root(n) with $tr(f) = tr(g)$. Then any index k with $w_k \neq x_k$ would imply that the kth component in $tr(f)$ and $tr(g)$ are different in contradiction to $tr(f) = tr(g)$. Thus $f = g$. Next we will show that for every $\varphi = (x_1, \ldots, x_n) \in$ End(P_n), there exists a rooted binary path