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Hamilton cycles and paths in vertex-transitive graphs—Current directions

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ABSTRACT

In this article current directions in solving Lovász's problem about the existence of Hamilton cycles and paths in connected vertex-transitive graphs are given.

1. Historical motivation

In 1969, Lovász [59] asked whether every finite connected vertex-transitive graph has a Hamilton path, that is, a simple path going through all vertices, thus tying together two seemingly unrelated concepts: traversability and symmetry of graphs. Arguably, however, the general problem of finding Hamilton paths and cycles in highly symmetric graphs may be much older, as it can be traced back to bell ringing, Gray codes and the knight's tour of a chessboard (see [1,21,38,48]). Lovász problem is, somewhat misleadingly, usually referred to as the Lovász conjecture, presumably in view of the fact that, after all these years, a connected vertex-transitive graph without a Hamilton path is yet to be produced. Moreover, only four connected vertex-transitive graphs (having at least three vertices) not having a Hamilton cycle are known to exist: the Petersen graph, the Coxeter graph, and the two graphs obtained from them by replacing each vertex with a triangle. All of these are cubic graphs, suggesting perhaps that no attempt to resolve the above problem can bypass a thorough analysis of cubic vertex-transitive graphs. However, none of these four graphs is a Cayley graph, that is, a vertex-transitive graph with a regular subgroup of automorphisms. This has led to a folklore conjecture that every connected Cayley graph possesses a Hamilton cycle. This problem, together with its Cayley graph variation, has spurred quite a bit of interest in the mathematical community producing, amongst other, conjectures and counterconjectures with regard to its truthfulness. Thomassen [18,82] conjectured that only finitely many connected vertex-transitive graphs without a Hamilton cycle exist, and Babai [15,16] conjectured that infinitely many such graphs exist. More precisely, he conjectured that there exists $\epsilon > 0$ such that there are infinitely many connected vertex-transitive graphs X with longest cycle of length at most $(1-\epsilon)|V(X)|$.

All in all, many articles directly and indirectly related to this subject (see [2–6,8–10,12–14,24,32,37,45,47,53,60,61,63,64,68,69,84–86] for some of the relevant references), have appeared in the literature, affirming the existence of such paths

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and, in some cases, even Hamilton cycles. For example, it is known that connected vertex-transitive graphs of order kp, where $k \leq 4$, (except for the Petersen graph and the Coxeter graph) of order p^j, where $j \leq 4$, and of order $2p^2$, where p is prime, contain a Hamilton cycle. Moreover, it is known that connected vertex-transitive graphs of order pq, where p and q are primes, admitting an imprimitive subgroup of automorphisms contain a Hamilton cycle. Also, a Hamilton path is known to exist in connected vertex-transitive graphs of order $5p$ and $6p$ (see [2,24,27,56,57,62,65,61,63,64,66,83]). As for a general vertex-transitive graph, the best known result is that of Babai, who has shown that a vertex-transitive graph on n vertices has a cycle of length at least $\sqrt{3n}$ [17].

Particular attention has been given to Cayley graphs. Nevertheless, most of the results proved thus far depend on various restrictions made either on the class or order of groups dealt with, or on the generating sets of Cayley graphs (see [10,13,14,43,44,52,78,86]). For example, one may easily see that connected Cayley graphs of abelian groups have a Hamilton cycle. Further, it is known that connected Cayley graphs of hamiltonian groups have a Hamilton cycle (see [13]), and that connected Cayley graphs of metacyclic groups with respect to the standard generating set have a Hamilton cycle (see [5]). Also, following a series of articles [37,53,60] it is now known that every connected Cayley graph of a group with a cyclic commutator subgroup of prime power order, has a Hamilton cycle. This result has later been generalized to connected vertex-transitive graphs whose automorphism groups contain a transitive subgroup with a cyclic commutator subgroup of prime power order, where the Petersen graph is the only counterexample [32]. But perhaps the biggest achievement on the subject is due to Witte (now Morris) who proved that a connected Cayley digraph of any p-group has a Hamilton cycle [86]. On the other hand, even for the class of dihedral groups the question remains open. The best result in this respect is due to Alspach [6] who proved that every connected Cayley graph on a generalized dihedral group of order divisible by 4 has a Hamilton cycle (see also [14]). Three other results of note are a theorem of Witte [84, Theorem 3.1] showing that every group G with minimal generating set of size d contains a generating set of size less than $4d^2$ such that the corresponding Cayley graph has a Hamilton cycle, a theorem of Pak and Radočić [78] showing that every group G has a generating set of size at most $\log_2 |G|$ for which the corresponding Cayley graph has a Hamilton cycle, and a theorem of Krivelevich and Sudakov [55] showing that for every $c > 0$ and large enough n, a Cayley graph formed by choosing a set of $c \log^5 n$ generators randomly from a given group of order n, almost surely has a Hamilton cycle. (For further results not explicitly mentioned or referred here see the survey articles [30,87]).

Various concepts/problems related to Hamilton cycles and paths in vertex-transitive graphs and digraphs, motivated by the original Lovász's question, such as Hamilton connectivity, Hamilton laceability, Hamilton decomposability and edge hamiltonicity, have been studied (see [7,13,25,26,58,88,73,74]). In this article, however, we will only consider the problem of existence of Hamilton paths and cycles in connected vertex-transitive graphs, hereafter refereed to as the *HPC problem*. Also, a graph possessing a Hamilton cycle is said to be *hamiltonian*.

The article is organized as follows. In Section 2 definitions, notation and some auxiliary results are introduced. In Section 3 the main strategies used thus far together with possible future directions in solving the HPC problem are given, in particular, the "lifting Hamilton cycles approach" (see Section 3.1) and the "Hamilton trees on surfaces approach" (see Section 3.2). In addition, the usefulness of general results on the existence of Hamilton cycles in graphs in connection to the HPC problem are also discussed.

2. Notation

Throughout this article graphs are finite, undirected and unless specified otherwise, connected. (In most cases the graphs are simple graphs, however in some instances multiple edges will be allowed.) Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\, X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A sequence $(u_0, u_1, u_2, \ldots, u_s)$ of distinct vertices in a graph is called an *s-arc* if u_i is adjacent to u_{i+1} for every $i \in \{0, 1, \ldots, s-1\}$. For $S \subseteq V(X)$ we let $X[S]$ denote the induced subgraph of X on S. By an n-cycle we shall always mean a cycle with n vertices. A subgroup $G \leq \mathrm{Aut}\, X$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. A subgroup $G \leq \mathrm{Aut}\, X$ is said to be *s-regular* if it acts regularly on the sets of s-arc of X. The graph X is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive, respectively. An arc-transitive graph is also called *symmetric*. Given a group G and a subset S of $G \setminus \{1\}$ such that $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ has vertex set G and edges of the form $\{g, gs\}$ for all $g \in G$ and $s \in S$. The symbol $\mathbb{Z}_r$ will denote both the cyclic group of order r and the ring of integers modulo r. In the latter case, $\mathbb{Z}_r^*$ will denote the multiplicative group of units of $\mathbb{Z}_r$. By D_{2n} we denote the dihedral group of order $2n$.

Given a transitive group G acting on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts from $\mathcal{B}$, called *blocks* of G, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise.

For a graph X and a partition $\mathcal{W}$ of $V(X)$, we let $X_{\mathcal{W}}$ be the associated *quotient graph* of X relative to $\mathcal{W}$, that is, the graph with vertex set $\mathcal{W}$ and edge set induced naturally by the edge set $E(X)$. Note that $X_{\mathcal{W}}$ may contain multiple edges. Given integers $k \geq 1$ and $n \geq 2$ we say that an automorphism of a graph is *(k, n)-semiregular* if it has k orbits of length n and no other orbit. In the case when $\mathcal{W}$ corresponds to the set of orbits of a semiregular automorphism $\rho \in \mathrm{Aut}\, X$, the symbol $X_{\mathcal{W}}$ will be replaced by X_ρ. One of the successful strategies in the search for Hamilton cycles in connected vertex-transitive graphs is based on an analysis singling out the structure of the quotient graphs of graphs in question relative to orbits of a semiregular automorphism (see Section 3.1).