

Hamilton connectedness and the partially square graphs

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Abstract

Let G be a κ -connected graph on n vertices. The *partially square graph* G^* of G is obtained by adding edges uv whenever the vertices u, v have a common neighbor x satisfying the condition $N_G(x) \subset N_G[u] \cup N_G[v]$. Clearly $G \subseteq G^* \subseteq G^2$, where G^2 is the square of G . In particular $G^* = G^2$ if G is quasi-claw-free (and claw-free). In this paper we prove that a κ -connected, ($\kappa \geq 3$) graph G is either hamiltonian-connected or the independence number of G^* is at least κ . As a consequence we answer positively two open questions. The first one by Ainouche and Kouider and the second one by Wu et al. As a by-product we prove that a quasi-claw-free (and hence a claw-free) graph satisfying the condition $\alpha(G^2) < \kappa$ is hamiltonian-connected.

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1. Introduction

We use Bondy and Murty [6] for terminology and notation not defined here and consider simple graphs $G = (V, E)$ only. If A, B are disjoint sets of V , we denote by $E(A, B)$ the set of edges with an end in A and the other in B and by $G[A]$ the subgraph induced by A .

For any vertex u of G , $N(u)$ denotes its neighborhood set and $N[u] = \{u\} \cup N(u)$. If $X \subset V$, we denote by $N_X(u)$ the set of vertices of X adjacent to u . Throughout $\alpha(H)$, $\kappa(H)$ stand, respectively, for the independence number and the connectivity of the graph H . For $1 \leq k \leq \alpha$, we put $I_k(H) = \{Y \mid Y \text{ is a } k\text{-independent set in } H\}$. With each pair (a, b) of nonadjacent vertices, we associate the set

$$J(a, b) := \{x \in N(a) \cap N(b) \mid N_G(x) \subset N_G[u] \cup N_G[v]\}.$$

The partially square graph G^* [3] of $G = (V, E)$ is the graph $(V, E \cup \{uv \mid J(u, v) \neq \emptyset\})$. Clearly $G \subseteq G^* \subseteq G^2$ where G^2 is the square of G . A graph is *quasi-claw-free graph* [2] if every pair (x, y) of vertices at distance two satisfies the condition $J(x, y) \neq \emptyset$. Equivalently, a graph G is quasi-claw-free if $G^* = G^2$. Clearly any quasi-claw-free graph is claw-free.

For each set $S \in I_k(G)$, $k \geq 1$ we adopt a partition of V by defining

$$S_i := \{u \mid |N_S(u)| = i\}, \quad s_i := |S_i|, \quad i = 0, \dots, k.$$

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Throughout S_i is defined with respect to G and not for G^* and obviously, we have $|N(S)| = \sum_{i=1}^k s_i$ and $\sigma_S := \sum_{x \in S} d(x) = \sum_{i=1}^k i s_i$.

A graph G is traceable if it contains a hamiltonian path, is 1-hamiltonian if $G - a$ is hamiltonian for any $a \in V(G)$, is edge-hamiltonian if any edge lies on a hamiltonian cycle and is hamiltonian-connected if for any pair (a, b) of vertices, G has a hamiltonian path with ends a, b .

Erdős and Chvátal established well-known sufficient conditions for a graph to be traceable, hamiltonian, 1-hamiltonian, edge-hamiltonian or hamiltonian-connected.

These conditions involve the independence number and the connectivity of G . More precisely they proved,

Theorem 1.1 (Chvátal and Erdős [7]). *A κ -connected graph G is*

- (1) *Traceable if $\alpha(G) \leq \kappa(G) + 1$.*
- (2) *Hamiltonian if $\alpha(G) \leq \kappa(G)$.*
- (3) *One-hamiltonian, one edge-hamiltonian and hamiltonian-connected if $\alpha(G) < \kappa(G)$.*

Ainouche and Kouider [3] extended the above results by showing that, except for a particular case, the results remain true if we replace G by G^* . They failed to prove that G is hamiltonian-connected if $\kappa(G) = 3$ and naturally they raised the following question.

Problem 1.2. Prove that a 3-connected graph is hamiltonian-connected if

$$\alpha(G^*) < 3.$$

Recently Wu et al. [9] extended a general sufficient condition given in [4], based on independent sets in G to independent sets in G^* . As Ainouche and Kouider, they also failed to cover the property of hamiltonian-connectedness for the case $\kappa(G) = 3$. This is not surprising because Wu et al. used the same proof technique as in [3]. A second problem considered in this paper is the following conjecture proposed in [9].

Problem 1.3. Prove that a 3-connected graph is hamiltonian-connected if $\sigma_S > n + s_3$ holds for every independent triple S in G^* .

With Theorems 1.4 and 1.7 we answer positively both problems. Some straightforward corollaries are also stated. Theorem 1.4 is known (see [3]) for $\kappa \geq 4$.

Theorem 1.4. *A κ -connected ($\kappa \geq 3$) graph G of order n is hamiltonian-connected if $\alpha(G^*) < \kappa$.*

Since $\alpha(G^*) = \alpha(G^2)$ if G is a quasi-claw-free (or a claw-free) graph, we easily get.

Corollary 1.5. *A κ -connected ($\kappa \geq 3$) quasi-claw-free (or a claw-free) graph G of order n is hamiltonian-connected if $\alpha(G^2) < \kappa$.*

Corollary 1.6. *A κ -connected ($\kappa \geq 3$) quasi-claw-free (or a claw-free) graph G of order n is hamiltonian-connected if $\sigma_S > n - \kappa$ holds for every $S \in I_{\kappa+1}(G)$.*

Theorem 1.7. *A 3-connected graph G of order n is hamiltonian-connected if $\sigma_S > n + s_3$ holds for every $S \in I_3(G^*)$.*

Theorem 1.7 extends an earlier result of Wei [8], where triples are considered in G . Under the same conditions of theorems 1.4 and 1.7, G is also 1-hamiltonian and edge-hamiltonian (see [3,9]). Moreover, Theorem 1.7 covers a large spectrum of new results dealing with the hamiltonian-connectedness property.

Corollary 1.8. *A 3-connected graph G of order n is hamiltonian-connected if every triple $S \in I_3(G^*)$ contains a vertex $u \in S$ such that $N(S \setminus \{u\}) + d(u) > n$.*

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