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Distance-residual subgraphs

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ABSTRACT

For a connected finite graph G and a subset V_0 of its vertex set, a distance-residual subgraph is a subgraph induced on the set of vertices at the maximal distance from V_0. Some properties and examples of distance-residual subgraphs of vertex-transitive, edge-transitive, bipartite and semisymmetric graphs are presented. The relations between the distance-residual subgraphs of product graphs and their factors are explored.

1. Introduction

Let $G = (V, E)$ be a connected finite graph and let V_0 be a nonempty subset of $V(G)$. Let $P(G, V_0) = \{V_0, V_1, \ldots, V_m\}$ denote a *distance partition* of G with respect to V_0, where

$$\begin{aligned} &V_0 \cup V_1 \cup \cdots \cup V_m = V(G), \\ &V_i \cap V_j = \emptyset, \quad \text{for } i \neq j, \\ &V_i \neq \emptyset, \quad \text{for all } i, \end{aligned} \tag{1}$$

and the sets V_i, $i = 1, 2, \ldots, m$, are defined recursively as

$$V_i := \{v \in V(G) \setminus (V_0 \cup V_1 \cup \cdots \cup V_{i-1}) \mid \exists u \in V_{i-1}, \{u, v\} \in E(G)\}.$$

Let $d(u, v)$ be the distance between the vertices u and v in G, i.e. the length of the shortest path between them. If G is not connected, then $d(u, v) := \infty$ for u and v that are from different connected components of G. Obviously, $d(v, v_i) \geq i$ for $v \in V_0$ and $v_i \in V_i$, and there exists a vertex $v_0 \in V_0$ for which $d(v_0, v_i) = i$. We also define the distance between a vertex $v \in G$ and a subgraph $R \subset G$ as $d(v, R) := \min_{r \in V(R)} d(v, r)$, and the distance between two subgraphs $R, S \subset G$ as $d(R, S) := \min_{s \in V(S)} d(s, R)$.

We are interested in induced subgraphs $G[V_i]$, defined by the distance sets V_i, particularly in the subgraph $R := G[V_0]$, called a *root*, and the subgraph $Res(G, R) := G[V_m]$ that we call a *distance-residual subgraph*. If the root is a single vertex v, $Res(G, G[v])$ is called a *vertex-residual subgraph* (or VR subgraph), and if the root R is induced by a pair of adjacent vertices, $Res(G, R)$ is called an *edge-residual subgraph* (or ER subgraph). We will assume that all graphs through the paper are nontrivial, simple, finite and in most cases connected. The standard notation for the well-known graphs will be used: K_n for complete graphs, $K_{m,n}$ for complete bipartite graphs, C_n for cycles, P_n for paths, and mK_n for the disjoint union of m complete graphs on n vertices.

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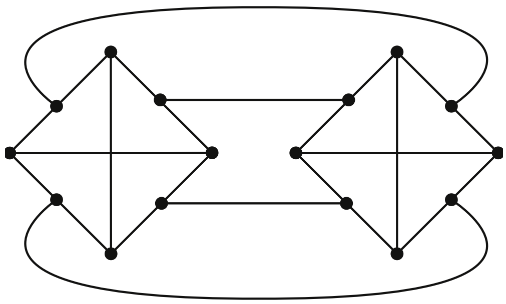

Fig. 1. A non-vertex-transitive cubic graph with isomorphic VR subgraphs.

The motivation for the definition of distance-residual subgraphs comes from the definition of the *distance degree sequence* [3], an ordered list where the *i*th element represents the number of vertices at distance *i* from the selected root. The distance degree sequence shows only the number of vertices at the distance *i* from the root, but we are interested also in the induced subgraphs on those vertices, especially on the set farthest from the root.

In what follows, we will often use the notation $\uplus$ to denote an induced graph on the vertices of two subgraphs of a given graph.

Definition 1.1. Let H and K be subgraphs of graph G. Then $H \uplus_G K := G[V(H) \cup V(K)]$. When there is no possibility of confusion, we will just write $H \uplus K$.

In the following section we present some properties of distance-residual subgraphs with focus on the vertex- and edge-transitive graphs, bipartite graphs, and semisymmetric graphs. In Section 3 we show how the distance-residual subgraphs of product graphs depend on the distance-residual subgraphs of their factors. We conclude the paper with some questions regarding distance-residual subgraphs and other distance related problems.

2. Properties

A distance-residual subgraph is defined only for connected graphs; however, any graph, connected or not, can be a distance-residual subgraph.

Theorem 2.1. *Let H and R be arbitrary graphs. Then there exists a connected graph G containing R as an induced subgraph such that H is isomorphic to $Res(G, R)$.*

Proof. Graph G is constructed with the join of H and R. Therefore, $V(G) = V(H) \cup V(R)$ and the edge set consists of all the original edges of H and R with an additional edge $\{u, r\}$ for each $v \in V(H)$ and each $r \in V(R)$. Clearly, the distance partition $P(G, V_0)$ is given by $V_0 = V(R)$ and $V_1 = V(H)$, hence $Res(G, R) = H$. □

2.1. Distance-residual subgraphs of vertex- and edge-transitive graphs

Lemma 2.2. *Let G be a connected vertex-transitive graph. Then all of its VR subgraphs are isomorphic, i.e. they are independent of the choice for the root. The converse is not true, not even for regular graphs.*

Proof. The first part of the lemma is obvious since the graph would not have a transitive automorphism group if the VR subgraphs were not isomorphic. The cubic graph in Fig. 1 proves that the converse is not true. It is constructed from two copies of K_4 by joining of their edges bordering the outer face, where a join of edges e and f represents an operation of inserting a new vertex u on e and v on f and adding a new edge uv to the graph [16]. All of its VR subgraphs are isomorphic (to K_1) but the cubic graph is not vertex-transitive since the automorphism that would map a vertex from K_4 to a vertex created by the joining of edges does not exist. □

If all the vertices of a graph G have the same distance degree sequence, G is called a *distance degree regular graph* (DDR). A graph that is DDR need not be vertex-transitive [3]. Note that the graph in Fig. 1 is DDR with the distance degree sequences $(1, 3, 6, 5, 1)$. Therefore even a DDR graph with isomorphic VR subgraphs need not be vertex-transitive.

Lemma 2.3. *Let G be a connected edge-transitive graph. Then all of its ER subgraphs are isomorphic, i.e. they are independent of the choice of the edge for the root. The converse is not true.*

Proof. The first part of the proof is similar to the proof of Lemma 2.2. A counterexample that confirms the second claim is the graph presented in Fig. 2, which has all of its ER subgraphs isomorphic (to K_1) but obviously it is not edge-transitive. □

Lemma 2.4. *Let G be a connected edge-transitive graph and $L(G)$ its line graph. Then all of the VR subgraphs of $L(G)$ are isomorphic.*