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Note

Factors and vertex-deleted subgraphs

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Dedicated to Professor Hikoe Enomoto on the occasion of his 60th birthday

Abstract

A relationship is considered between an *f*-factor of a graph and that of its vertex-deleted subgraphs. Katerinis [Some results on the existence of $2n$-factors in terms of vertex-deleted subgraphs, Ars Combin. 16 (1983) 271–277] proved that for even integer k, if $G-x$ has a k-factor for each $x \in V(G)$, then G has a k-factor. Enomoto and Tokuda [Complete-factors and *f*-factors, Discrete Math. 220 (2000) 239–242] generalized Katerinis' result to *f*-factors, and proved that if $G-x$ has an *f*-factor for each $x \in V(G)$, then G has an *f*-factor for an integer-valued function *f* defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even. In this paper, we consider a similar problem to that of Enomoto and Tokuda, where for several vertices x we do not have to know whether $G-x$ has an *f*-factor. Let G be a graph, X be a set of vertices, and let *f* be an integer-valued function defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even, $|V(G)-X| \geqslant 2$. We prove that if $\sum_{x\in X} \deg_G(x) \leqslant 2|V(G)-X|-1$ and if $G-x$ has an *f*-factor for each $x \in V(G)-X$, then G has an *f*-factor. Moreover, if G excludes an isolated vertex, then we can replace the condition $\sum_{x\in X} \deg_G(x) \leqslant 2|V(G)-X|-1$ with $\sum_{x\in X} \deg_G(x) \leqslant 2|V(G)-X|+|X|-3$. Furthermore the condition will be $\sum_{x\in X} \deg_G(x) \leqslant 2|V(G)-X|-1$ when $|X|=1$.

Keywords: *f*-Factor; Odd component; Vertex-deleted subgraph

1. Introduction

We consider finite undirected graphs which may have *loops* and *multiple edges*. Let G be a graph. For $x \in V(G)$, we denote by $\deg_G(x)$ the *degree* of x in G. The set of *neighbours* of $x \in V(G)$ is denoted by $N_G(x)$. For disjoint subsets S and T of $V(G)$, we denote by $e_G(S,T)$ the number of the edges joining S and T. If S or T is a singleton set $\{x\}$, we write x instead of $\{x\}$. For example, we write $e_G(x,T)$ instead of $e_G(\{x\},T)$. Let *f* be an integer-valued function defined on $V(G)$. A spanning subgraph F of G such that $\deg_F(x)=f(x)$ for each $x \in V(G)$ is called an *f-factor* of G. If *f* is a constant function taking a value k, an *f*-factor is called a *k-factor*. When no fear of confusion arises, we often identify an *f*-factor with its edge set. In other words, for a graph G, we say that a subset F of $E(G)$ is an *f*-factor if $(V(G),F)$ is an *f*-factor.

Concerning a relationship between a k-factor of a graph and that of vertex-deleted subgraphs, Katerinis proved the following theorem.

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Theorem 1 (*Katerinis [2]*). *Let G be a graph of order at least two, and k be a positive integer. If $G-x$ has a 2k-factor for each $x \in V(G)$, then G has a 2k-factor.*

The above theorem was extended to an *f*-factor by Enomoto and Tokuda.

Theorem 2 (*Enomoto and Tokuda [1]*). *Let G be a graph of order at least two, and let f be an integer-valued function defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even. If $G-x$ has an f-factor for each $x \in V(G)$, then G has an f-factor.*

We remark that Enomoto and Tokuda actually proved a stronger statement than Theorem 2. Also note that if G has an *f*-factor, then $\sum_{x\in V(G)} f(x)$ is even.

In Theorems 1 and 2, every vertex x is examined to satisfy the condition that $G-x$ has an *f*-factor. Then a question arises whether all the vertices are necessarily examined. Motivated by the above question, we consider the same type of theorems, but with some "unexamined vertices". And we prove the following results.

Theorem 3. *Let G be a graph, and let f be an integer-valued function defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even. Let X be a subset of $V(G)$, $|V(G)-X|\geqslant 2$. Suppose $\sum_{x\in X}\deg_G(x)\leqslant 2|V(G)-X|-1$. If $G-x$ has an f-factor for each $x\in V(G)-X$, then G has an f-factor.*

Theorem 4. *Let G be a graph not to include an isolated vertex, and let f be an integer-valued function defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even. Let X be a subset of $V(G)$, $|V(G)-X|\geqslant 2$. Suppose $\sum_{x\in X}\deg_G(x)\leqslant 2|V(G)-X|+|X|-3$. When $|X|=1$, suppose $\sum_{x\in X}\deg_G(x)\leqslant 2|V(G)-X|-1$. If $G-x$ has an f-factor for each $x\in V(G)-X$, then G has an f-factor.*

In order to prove Theorems 3 and 4, we use Tutte's *f*-Factor Theorem. Let G be a graph. For disjoint subsets S and T of $V(G)$, we define $\delta_G(S,T)$ by

$$\delta_G(S,T)=\sum_{x\in S} f(x)+\sum_{x\in T}(\deg_{G-S}(x)-f(x))-h_G(S,T),$$

where $h_G(S,T)$ is the number of components C of $G-(S\cup T)$ such that $\sum_{x\in V(C)} f(x)+e_G(V(C),T)$ is odd. These components are called *odd* components. We denote by $\mathscr{H}_G(S,T)$ the set of the odd components, i.e., $|\mathscr{H}_G(S,T)|=h_G(S,T)$.

Theorem 5 (*Tutte [3]*). *Let G be a graph, and let f be an integer-valued function defined on $V(G)$. Then*

(1) $\delta_G(S,T)\equiv\sum_{x\in V(G)} f(x) \pmod 2$ *for each disjoint subsets S and T of $V(G)$, and*
(2) *G has an f-factor if and only if $\delta_G(S,T)\geqslant 0$ for each pair of disjoint subsets S and T of $V(G)$.*

2. Proof of Theorem 3

Assume that there exists a graph G, that $X\subset V(G)$ satisfying $\sum_{x\in X}\deg_G(x)\leqslant 2|V(G)-X|-1$, and that an integer-valued function f defined on $V(G)$ with $\sum_{x\in V(G)} f(x)$ even such that G has no *f*-factor while $G-x$ has an *f*-factor for each $x\in V(G)-X$. By Theorem 2, we may assume that $X\neq\emptyset$.

Since G has no *f*-factor and $\sum_{x\in V(G)} f(x)$ is even, we obtain disjoint subsets S and T of $V(G)$ with $\delta_G(S,T)\leqslant -2$ by Theorem 5. Let $U=V(G)-(S\cup T)$. For $x\in V(G)$, let $\mu(x)$ be the number of components H in $\mathscr{H}_G(S,T)$ such that each component of $H-x$ is not in $\mathscr{H}_{G-x}(S-x,T-x)$.

Suppose $x\in S$. By the definition of $\delta_G(S,T)$, $\delta_{G-x}(S-x,T)\leqslant\delta_G(S,T)\leqslant -2$, and we have $x\in X$ and $S\subset X$.

Suppose $x\in U$. Let C be the component of $G-(S\cup T)$ that contains x. Then by the definition of an odd component, $\mathscr{H}_G(S,T)-\{C\}\subseteq\mathscr{H}_{G-x}(S,T)$, and hence $\mu(x)\leqslant 1$. Therefore, $\delta_{G-x}(S,T)\leqslant\delta_G(S,T)+1\leqslant -1$, and we obtain $x\in X$ and $U\subset X$.

Suppose $x\in T-X$. By the definition of an odd component,

$$e_G(x,U)\geqslant h_G(S,T)-h_{G-x}(S,T-x). \tag{1}$$