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Stability results for uniquely determined sets from two directions in discrete tomography

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ABSTRACT

In this paper we prove several new stability results for the reconstruction of binary images from two projections. We consider an original image that is uniquely determined by its projections and possible reconstructions from slightly different projections. We show that for a given difference in the projections, the reconstruction can only be disjoint from the original image if the size of the image is not too large. We also prove an upper bound for the size of the image given the error in the projections and the size of the intersection between the image and the reconstruction.

1. Introduction

Discrete tomography is concerned with problems such as reconstructing binary images on a lattice from given projections in lattice directions [6]. Each point of a binary image has a value equal to zero or one. The line sum of a line through the image is the sum of the values of the points on this line. The projection of the image in a certain lattice direction consists of all the line sums of the lines through the image in this direction.

Several problems related to the reconstruction of binary images from two or more projections have been described in the literature [6,7]. Already in 1957, Ryser gave an algorithm to reconstruct binary images from their horizontal and vertical projections and characterised the set of projections that correspond to a unique binary image [11]. For any set of directions, it is possible to construct images that are not uniquely determined by their projections in those directions [6, Theorem 4.3.1]. The problem of deciding whether an image is uniquely determined by its projections and the problem of reconstructing it are NP-hard for any set of more than two directions [4].

Aside from various interesting theoretical problems, discrete tomography also has applications in a wide range of fields. The most important are electron microscopy [8] and medical imaging [5,13], but there are also applications in nuclear science [9,10] and various other fields [12,15].

An interesting problem in discrete tomography is the stability of reconstructions. Even if an image is uniquely determined by its projections, a very small error in the projections may lead to a completely different reconstruction [1,3]. Alpers et al. [1,2] showed that in the case of two directions a total error of at most 2 in the projections can only cause a small difference in the reconstruction. They also proved a lower bound on the error if the reconstruction is disjoint from the original image.

In this paper we improve this bound, and we resolve the open problem of stability with a projection error greater than 2.

2. Notation and statement of the problems

Let F_1 and F_2 be two finite subsets of $\mathbb{Z}^2$ with characteristic functions χ_1 and χ_2. (That is, $\chi_h(x, y) = 1$ if and only if $(x, y) \in F_h$, $h \in \{1, 2\}$.) For $i \in \mathbb{Z}$, we define *row i* as the set $\{(x, y) \in \mathbb{Z}^2 : x = i\}$. We call i the index of the row. For $j \in \mathbb{Z}$,

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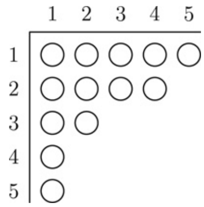

Fig. 1. A uniquely determined set with the assumed row and column ordering.

we define *column j* as the set $\{(x, y) \in \mathbb{Z}^2 : y = j\}$. We call j the index of the column. Following matrix notation, we use row numbers that increase when going downwards and column numbers that increase when going to the right.

The *row sum* $r_i^{(h)}$ is the number of elements of F_h in row i, that is $r_i^{(h)} = \sum_{j\in\mathbb{Z}} \chi_h(i, j)$. The *column sum* $c_j^{(h)}$ of F_h is the number of elements of F_h in column j, that is $c_j^{(h)} = \sum_{i\in\mathbb{Z}} \chi_h(i, j)$. We refer to both row and column sums as the *line sums* of F_h.

Throughout this paper, we assume that F_1 is uniquely determined by its row and column sums. Such sets were studied by, among others, Ryser [11] and Wang [14]. Let a be the number of rows and b the number of columns that contain elements of F_1. We renumber the rows and columns such that we have

$$r_1^{(1)} \geq r_2^{(1)} \geq \cdots \geq r_a^{(1)} > 0,$$
$$c_1^{(1)} \geq c_2^{(1)} \geq \cdots \geq c_b^{(1)} > 0,$$

and such that all elements of F_2 are contained in rows and columns with positive indices. By [14, Theorem 2.3] we have the following property of F_1 (see Fig. 1):

- in row i the elements of F_1 are precisely the points $(i, 1), (i, 2), \ldots, (i, r_i^{(1)})$,
- in column j the elements of F_1 are precisely the points $(1, j), (2, j), \ldots, (c_j^{(1)}, j)$.

We will refer to this property as the *triangular shape* of F_1.

Everywhere except in Section 6 we assume that $|F_1| = |F_2|$. Note that we do not assume F_2 to be uniquely determined.

As F_1 and F_2 are different and F_1 is uniquely determined by its line sums, F_2 cannot have exactly the same line sums as F_1. Define the *difference* or *error in the line sums* as

$$\sum_{j\geq 1} |c_j^{(1)} - c_j^{(2)}| + \sum_{i\geq 1} |r_i^{(1)} - r_i^{(2)}|.$$

As in general $|t - s| \equiv t + s \bmod 2$, the above expression is congruent to

$$\sum_{j\geq 1} \left(c_j^{(1)} + c_j^{(2)}\right) + \sum_{i\geq 1} \left(r_i^{(1)} + r_i^{(2)}\right) \equiv 2|F_1| + 2|F_2| \equiv 0 \bmod 2,$$

hence the error in the line sums is always even. We will denote it by 2α, where α is a positive integer.

For notational convenience, we will often write p for $|F_1 \cap F_2|$.

We consider two problems concerning stability.

Problem 1. *Suppose $F_1 \cap F_2 = \emptyset$. How large can $|F_1|$ be in terms of α?*

Alpers et al. [2, Theorem 29] proved that $|F_1| \leq \alpha^2$. They also showed that there is no constant c such that $|F_1| \leq c\alpha$ for all F_1 and F_2. In Section 4 of this paper we will prove the new bound $|F_1| \leq \alpha(1 + \log \alpha)$ and show that this bound is asymptotically sharp.

Problem 2. *How small can $|F_1 \cap F_2|$ be in terms of $|F_1|$ and α, or, equivalently, how large can $|F_1|$ be in terms of $|F_1 \cap F_2|$ and α?*

Alpers ([1, Theorem 5.1.18]) showed in the case $\alpha = 1$ that

$$|F_1 \cap F_2| \geq |F_1| + \frac{1}{2} - \sqrt{2|F_1| + \frac{1}{4}}.$$

This bound is sharp: if $|F_1| = \frac{1}{2}n(n + 1)$ for some positive integer n, then there exists an example for which equality holds. A similar result is stated in [2, Theorem 19].