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Mixed graph edge coloring[☆]

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ABSTRACT

We are interested in coloring the edges of a mixed graph, i.e., a graph containing unoriented and oriented edges. This problem is related to a communication problem in job-shop scheduling systems. In this paper we give general bounds on the number of required colors and analyze the complexity status of this problem. In particular, we provide $\mathcal{NP}$-completeness results for the case of outerplanar graphs, as well as for 3-regular bipartite graphs (even when only 3 colors are allowed, or when 5 colors are allowed and the graph is fully oriented). Special cases admitting polynomial-time solutions are also discussed.

1. Introduction

A mixed graph $G_M = (V, U, E)$ is a graph containing unoriented edges (set E) as well as oriented edges (set U), referred to as *arcs*. This notion was first introduced in [18].

1.1. Related work: Vertex coloring of mixed graphs

Vertex coloring problems in mixed graphs have applications in scheduling, where disjunctive and precedence constraints have to be taken into account simultaneously. In particular, two variants of the problem have been given most attention in the literature (see for instance [18,1,2,4–6,8,11,13–17,19]).

In the first problem, simply called *mixed graph vertex coloring*, the goal is to color the vertices of a mixed graph with a given number of colors, such that any two adjacent vertices get different colors, and for any arc (x, y), the color of x must be strictly smaller than the color of y. Notice that a solution only may exist if the oriented part of the mixed graph contains no oriented circuit. Furthermore, the mixed graph vertex coloring problem is a generalization of the usual coloring problem in unoriented graphs, and it has been shown to be $\mathcal{NP}$-complete even in planar cubic bipartite graphs (see [14]). In [4–6] polynomial algorithms are given for the cases of mixed trees and mixed series-parallel graphs. Bounds on the mixed chromatic number (i.e., the smallest integer for which the mixed graph admits a coloring) are presented in [15]. Finally,

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in [16,17] the unit-time job-shop problem is considered via mixed graph coloring, and branch-and-bound algorithms are given and tested on randomly generated mixed graphs.

In the second problem, known as *weak mixed graph vertex coloring*, we have the previous constraints, but we allow vertices linked by an arc to get the same color, i.e., for any arc (x, y) the color of x must be smaller than or equal to the color of y. In general, the previously described mixed graph vertex coloring problem can be treated as a special case of the weak version. Weak mixed graph vertex coloring is also known to be $\mathcal{NP}$-complete in planar cubic bipartite graphs [14]. Bounds on the mixed chromatic number are presented in [15]. In [19] some algorithms for calculating the exact value of the weak mixed chromatic number of graphs of order $n \leq 40$ and upper bounds for mixed graphs of order larger than 40, are presented.

1.2. Problem formulation and motivation: Edge coloring of mixed graphs

In this paper, we shall consider an edge coloring problem in mixed graphs. More precisely, we want to color the edges of a mixed graph $G_M = (V, U, E)$ such that any two adjacent edges (oriented and unoriented) get different colors and for any two adjacent arcs $e, e' \in U$ forming a directed path (e, e'), the color of e must be strictly less than the color of e'. Such a coloring will be called a *mixed graph edge coloring*. If only k colors are available, we call it a *mixed graph edge k-coloring*. The smallest integer k for which a graph G_M admits a mixed graph edge k-coloring will be called the *mixed chromatic index* of G_M and denoted by $q_M(G_M)$. Notice that for a solution to the mixed graph edge coloring problem to exist, the mixed graph must not contain any oriented circuit. Throughout the rest of the paper we shall assume that this is true. To the best of our knowledge, the mixed graph edge coloring problem has not been studied before; some basic properties and the motivation of the problem are discussed below.

Mixed graph edge coloring can be treated as a special case of mixed graph vertex coloring. For a mixed graph $G_M = (V, U, E)$, we define its *mixed line graph* $L(G_M)$ as the mixed graph having vertex set $U \cup E$, arcs (e, e') connecting all pairs of elements $e, e' \in U$ such that arc e ends at the start-vertex of arc e', and unoriented edges connecting all the remaining pairs of elements of $U \cup E$ which share at least one vertex. By analogy to the correspondence between an edge coloring of an undirected graph and a vertex coloring of its line graph, it is evident that a mixed graph edge coloring of G_M is proper if, and only if, the corresponding labeling of the vertices of $L(G_M)$ is a proper mixed graph vertex coloring.

Edge coloring of undirected graphs is often used to model certain job-shop scheduling instances consisting of unit-time tasks assigned to specific pairs of processors [10]. In the case of mixed graphs, it is convenient to look upon an arc from a node u to a node v as a unit-time data transmission process from u to v, requiring the cooperation of processors u and v, which cannot simultaneously perform other tasks. Thus, a correct coloring of the directed arcs of the graph corresponds to a scheduling in which each node first successively receives input data from all incoming arcs, next uses all the collected data for local computations (assumed to be instantaneous), and finally successively sends the output data along all its outgoing arcs. The undirected edges of the mixed graph, which only appear in some considerations, correspond to possibly unrelated two-processor tasks performed in the system, such as mutual self-testing of processors.

1.3. Definitions and notions

Let $G_M = (V, U, E)$ be a mixed graph. We shall denote by $l(G_M)$ the number of oriented edges on a longest directed path in G_M, and by $\Delta(G_M)$ the maximum degree of a vertex in G_M, i.e., the maximum number of edges (unoriented and oriented) incident to a same vertex $v \in V$. The outer degree of a vertex v, denoted by $\deg_{out}(v)$, is defined as the number of oriented edges (arcs) having v as the start-vertex; analogously, the inner degree of v, denoted by $\deg_{in}(v)$, is defined as the number of oriented edges (arcs) with v as the end-vertex. Finally, the *inrank* of a vertex v, denoted by $in(v)$, is the length, i.e., the number of arcs, of a longest directed path ending at v. For all graph theoretical terms not defined here, the reader is referred to [3].

1.4. Contribution and outline of the paper

The rest of the paper is organized as follows:

- In Section 2 we propose lower and upper bounds on the value of the mixed chromatic index of a graph G_M, expressed in terms of $l(G_M)$ and $\Delta(G_M)$. Interestingly, for all cases which are not equivalent to undirected edge coloring ($l(G_M) > 1$), these bounds turn out to be tight, even when G_M is a mixed tree having only oriented edges.
- In Section 3 we study the complexity of the problem *MGEC* of determining if $q_M(G_M) \leq k$ for a mixed graph G_M and integer k given at input. The problem turns out to be $\mathcal{NP}$-complete even if G_M is a bipartite outerplanar graph (Section 3), but it admits a polynomial solution for trees (Section 3.2).
- In Section 4 we consider the edge coloring problem for fully directed graphs (i.e., mixed graphs without unoriented edges, $E = \emptyset$), which appears to be of particular significance from a practical viewpoint. In this case, *MGEC* is shown to be $\mathcal{NP}$-complete even in 3-regular bipartite graphs when allowing 5 colors.
- Finally, we consider the complexity of the *MGEC* problem for bounded values of number of available colors (Section 5). By a generic argument, the problem is then solvable in polynomial time for partial k-trees, but turns out to be $\mathcal{NP}$-complete even in 3-regular planar bipartite mixed graphs when allowing 3 colors.