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On double bound graphs and forbidden subposets

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ABSTRACT

For a poset $P = (X, \leq_P)$, the *double bound graph* (DB-graph) of P is the graph $DB(P) = (X, E_{DB(P)})$, where $xy \in E_{DB(P)}$ if and only if $x \neq y$ and there exist $n, m \in X$ such that $n \leq_P x, y \leq_P m$. We obtain that for a subposet Q of a poset P, Q is an (n, m)-subposet of P if and only if $DB(Q)$ is an induced subgraph $DB(P)$. Using this result, we show some characterizations of split double bound graphs, threshold double bound graphs and difference double bound graphs in terms of (n, m)-subposets and double canonical posets.

1. Introduction

In this paper, we deal with graphs on finite posets. For a poset P and $x \in V(P)$, $L_P(x) = \{u \in V(P); u <_P x\}$ and $U_P(x) = \{u \in V(P); x <_P u\}$. For a poset P and $S \subseteq V(P)$, $L_P(S) = \bigcup_{x\in S} L_P(x)$ and $U_P(S) = \bigcup_{x\in S} U_P(x)$. Furthermore, $\mathrm{Max}(P)$ is the set of maximal elements of P and $\mathrm{Min}(P)$ is the set of minimal elements of P. For a poset P, a poset Q is a *subposet* of P if and only if (1) $V(Q) \subseteq V(P)$ and, (2) for all $x, y \in V(Q)$, $x \leq_Q y$ in Q only if $x \leq_P y$ in P, and a poset Q is an *induced subposet* of P if and only if (1) $V(Q) \subseteq V(P)$ and, (2) for all $x, y \in V(Q)$, $x \leq_Q y$ in Q if and only if $x \leq_P y$ in P. For a poset P and $S \subseteq V(P)$, $\langle S\rangle_P$ is the induced subposet on S.

A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. In some cases we consider that a clique is a maximal complete subgraph. In the same way, we occasionally abuse terms of induced subgraphs and the vertex set of induced subgraphs, especially for complete subgraphs. A family $\mathcal{D}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$, there exists $D \in \mathcal{D}$ such that $u, v \in D$. A family $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ is an *edge clique cover* of G if each D_i is a clique of G and $\mathcal{D}$ edge covers G.

For a graph G and $S \subseteq V(G)$, $\langle S\rangle_G$ is the induced subgraph on S. For a graph G and $v \in V(G)$, $N_G(v) = \{u; uv \in E(G)\}$.

For a poset $P = (X, \leq_P)$, the *double bound graph* (*DB-graph*) of P is the graph $DB(P) = (X, E_{DB(P)})$, where $xy \in E_{DB(P)}$ if and only if $x \neq y$ and there exist $n, m \in X$ such that $n \leq_P x, y \leq_P m$. McMorris and Zaslavsky [7] introduced this concept.

Diny [2] gives a characterization of double bound graphs as follows. For a graph G with two disjoint independent subsets M_G and N_G of $V(G)$ and $v \in V(G) - (M_G \cup N_G)$, define the set $U_{M_G}(v) = \{u \in M_G; uv \in E(G)\}$, $L_{N_G}(v) = \{u \in N_G; uv \in E(G)\}$.

Theorem 1 (*Diny [2]*). *A graph G is a DB-graph if and only if there exist a family $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ of complete subgraphs of G and disjoint independent subsets M_G and N_G of G such that:*

(1) *$\mathcal{D}$ edge covers G, and*
(2) *For each D_i, there exist $m_i \in M_G$ and $n_i \in N_G$ such that $\{m_i, n_i\} \subseteq D_i$ and $\{m_i, n_i\} \not\subseteq D_j$ for all $j \neq i$, and*
(3) *For each $v \in V(G) - (M_G \cup N_G)$, $|U_{M_G}(v)| \times |L_{N_G}(v)|$ equals the number of cliques of $\mathcal{D}$ containing v.*

Furthermore, a family $\mathcal{D}$ is the unique, minimal edge covering family of cliques in G.

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In the proof of this result, Diny use a fact that intervals of a poset correspond to complete subgraphs of a DB-graph. For a DB-graph G, an edge clique cover $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ satisfying the conditions of Theorem 1 is called a *DB edge clique cover*. For a DB edge clique cover $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$, M_G is a *upper kernel* $UK_{DB}(G)$ of G and N_G is a *lower kernel* $LK_{DB}(G)$ of G. We know a fact that for a corresponding poset P of a DB-graph G, $UK_{DB}(G)$ corresponds to the set $\mathrm{Max}(P)$ of all maximal elements of P and $LK_{DB}(G)$ corresponds to the set $\mathrm{Min}(P)$ of all minimal elements of P. In the following we consider a DB-graph with a fixed upper kernel $UK_{DB}(G)$ and a fixed lower kernel $LK_{DB}(G)$.

For a DB-graph G, $\mathcal{P}_{DB}(G) = \{P\ ;\ DB(P) \cong G, \mathrm{Max}(P) = UK_{DB}(G)$ and $\mathrm{Min}(P) = LK_{DB}(G)\}$. For a poset P, the *double canonical poset* of P is the poset $d\_can(P) = (V(P), \leq_{d\_can(P)})$, where $x \leq_{d\_can(P)} y$ if and only if (1) $x \in \mathrm{Min}(P)$, $y \in \mathrm{Max}(P)$ and $x \leq_P y$, or (2) $x \notin \mathrm{Max}(P) \cup \mathrm{Min}(P)$, $y \in \mathrm{Max}(P)$ and $x \leq_P y$, or (3) $x \in \mathrm{Min}(P)$, $y \notin \mathrm{Max}(P) \cup \mathrm{Min}(P)$ and $x \leq_P y$, or (4) $x = y$. We also introduce the double canonical poset $d\_can(G)$ of a DB-graph G, where $V(d\_can(G)) = V(G)$, $x \leq_{d\_can(G)} y$ if and only if (1) $x \in LK_{DB}(G)$, $y \in UK_{DB}(G)$ and $xy \in E(G)$, or (2) $x \notin LK_{DB}(G) \cup UK_{DB}(G)$, $y \in UK_{DB}(G)$ and $xy \in E(G)$, or (3) $x \in LK_{DB}(G)$, $y \notin LK_{DB}(G) \cup UK_{DB}(G)$ and $xy \in E(G)$, or (4) $x = y$. For a DB-graph G, all posets in $\mathcal{P}_{DB}(G)$ have the same double canonical poset by Theorem 1. Thus for a DB-graph G and each poset $P \in \mathcal{P}_{DB}(G)$, $d\_can(G) \cong d\_can(P)$ and $d\_can(G)$ is the minimum poset of $\mathcal{P}_{DB}(G)$. So properties of $d\_can(G)$ is key properties on DB-graphs.

In [5,8,9], first Scott, then we deal with upper bound graphs in terms of forbidden subgraphs and forbidden subposets. In this paper, we deal with double bound graphs in terms of forbidden subgraphs and forbidden subposets.

There exists an induced subposet Q of a poset P such that $DB(Q)$ is not an induced subgraph of $DB(P)$. So we introduce a concept of (n, m)-subposets. For a poset P, a poset Q is an *(n, m)-subposet* of P if and only if Q is a subposet of P with additional property that if $x, y \in V(Q)$ and $n \leq_P x, y \leq_P m$ for some $n, m \in V(P)$, then there exist $n', m' \in V(Q)$ with $n' \leq_Q x, y \leq_Q m'$. Iwai, Ogawa and Tsuchiya [5] introduce this concept. We already know the following result.

Proposition 2 (*Iwai, Ogawa and Tsuchiya [5]*). *Let P be a poset and Q be a subposet of P.Q is an (n, m)-subposet of P if and only if $DB(Q)$ is an induced subgraph of $DB(P)$.*

For a triangle-free graph, the edge clique cover is unique. So in a corresponding poset P of a DB-graph G, there exist small number of (n, m)-subposets of P which correspond to a given triangle-free induced subgraph of G. Based on this fact, Iwai, Ogawa and Tsuchiya deal with chordal double bound graphs [5]. In this paper we deal with split double bound graphs, threshold double bound graphs and difference double bound graphs in terms of (n, m)-subposets and double canonical posets.

2. Split double bound graphs

First we consider split DB-graphs. A graph G is a *split graph* if its vertices can be partitioned into an independent set and the vertex set of a complete subgraph. Földes and Hammer [3] give a characterization of split graphs.

Q_n is a poset such that (1) $V(Q_n) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \cup \{\beta_1, \beta_2, \ldots, \beta_n\}$, and (2) $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ are antichains of Q_n, and (3) $\alpha_i \leq \beta_j$ if and only if (i) $j = i$ or $j = i + 1$ for $i \neq n$ and (ii) $j = n$ or $j = 1$ for $i = n$. For posets P and Q, $P \oplus Q$ is the poset, where $V(P \oplus Q) = V(P) \cup V(Q)$ and $x \leq_{P\oplus Q} y$ if and only if (1) $x, y \in V(P)$ and $x \leq_P y$, (2) $x, y \in V(Q)$ and $x \leq_Q y$, or (3) $x \in V(P)$ and $y \in V(Q)$.

Theorem 3 (*Iwai, Ogawa and Tsuchiya [5]*). *Let P be a poset. Then DB(P) contains $C_s (s \geq 4)$ as an induced subgraph if and only if (1) the induced subposet $\langle \mathrm{Max}(P) \cup \mathrm{Min}(P)\rangle_P$ contains $Q_m (m \geq 2)$ as an induced subposet, or (2) $d\_can(P)$ contains $\{\delta\} \oplus Q_n$ or $Q_n \oplus \{\delta\} (n \geq 4)$ as an (n, m)-subposet.*

Theorem 4 (*Földes and Hammer [3]*). *Let G be a graph. The following statements are equivalent.*

(1) *G is a split graph.*
(2) *G does not contain $2K_2$, C_4 and C_5 as induced subgraphs.*

Using these results, we also obtain the following result on split DB-graphs. The posets shown in Fig. 2 are denoted by P_{2K_2} and $P_{K_{2,2}} (=Q_2)$.

Theorem 5. *Let G be a connected DB-graph with $UK_{DB}(G)$ and $LK_{DB}(G)$. The following statements are equivalent.*

(1) *G is a split* DB-*graph.*
(2) *$d\_can(G)$ does not contain the poset P_{2K_2} and the poset $P_{K_{2,2}}$ as (n, m)-subposets.*

Proof. (1) $\Rightarrow$ (2) Let G be a DB-graph and $d\_can(G)$ has P_{2K_2} or $P_{K_{2,2}}$ as an (n, m)-subposet. Then $DB(P_{2K_2})$ is $2K_2$ and $DB(P_{K_{2,2}})$ is C_4. Thus G has $2K_2$ or C_4 as an induced subgraph by Proposition 2. Therefore G is not a split DB-graph by Theorem 4.

(2) $\Rightarrow$ (1) We assume that G is a DB-graph and not a split graph. By Theorem 4, G contains one of $2K_2$, C_4 or C_5 as an induced subgraph. First we assume that G contains $2K_2$ as an induced subgraph. Let $V(2K_2) = \{v_1, u_1, v_2, u_2\}$ and $E(2K_2) = \{v_1u_1, v_2u_2\}$. By the definition of $2K_2$, $v_1 \parallel v_2$, $v_1 \parallel u_2$, $u_1 \parallel v_2$ and $u_1 \parallel u_2$ in $d\_can(G)$. Since G is a DB-graph, there exist cliques D_1 and D_2 of a DB-edge clique cover of G such that $v_1, u_1 \in D_1$ and $v_2, u_2 \in D_2$, and also exist $\{m_1, n_1\} \subseteq D_1$