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Non-planar core reduction of graphs☆

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Abstract

We present a reduction method that reduces a graph to a smaller core graph which behaves invariant with respect to non-planarity measures like crossing number, skewness, coarseness, and thickness. The core reduction is based on the decomposition of a graph into its triconnected components and can be computed in linear time. It has applications in heuristic and exact optimization algorithms for the non-planarity measures mentioned above. Experimental results show that this strategy reduces the number of edges by 45% in average for a widely used benchmark set of graphs.

Keywords: Preprocessing; Graph reduction; Crossing number; Skewness; Coarseness; Thickness

1. Introduction

Graph drawing is concerned with the problem of rendering a given graph on the two-dimensional plane so that the resulting drawing is as readable as possible. Objective criteria for the readability of a drawing depend mostly on the application domain, but achieving a drawing without edge crossings is in general a primary objective. Such a drawing is called a *planar* drawing. However, it is well known that not every graph can be drawn without edge crossings. The famous theorem by Kuratowski [38] shows that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5. Such a subdivision is commonly referred to as a *Kuratowski subdivision*.

If a graph G is not planar, the following question arises naturally: How far away is G from planarity? For that reason, various measures for non-planarity have been proposed; see also [39] for a survey. The most prominent measure is the *crossing number* $\nu(G)$ which asks for the minimum number of crossings in any drawing of G. The crossing number problem goes back to Turán who started to examine this problem during the Second World War (see, e.g., [26,43] for the whole story). More precisely, he was interested in the crossing number of the complete bipartite graph. At the beginning of the 1950s, Zarankiewicz [49] and Urbanik [46] independently claimed to have a solution for this problem, but Guy [26] showed that their proof contained an error. Until now, the crossing number of both

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the complete bipartite graph and the complete graph is still only conjectured. A comprehensive bibliography on the crossing number problem is maintained by Vrťo [47].

Another measure for the non-planarity of a graph is the *skewness* $\mu(G)$ which is the minimum number of edges that have to be removed from G in order to obtain a planar graph. This is computationally equivalent to the *maximum planar subgraph* problem that asks for a planar subgraph of G with the maximum number of edges. Probably the first reference on the skewness is Kotzig [37], where he gives a formula for the skewness of the complete graph and the complete bipartite graph.

Both the crossing number and the skewness problem have drawn much attention in the literature, whereas only a few references can be found for two further non-planarity measures: the coarseness and the thickness. The *thickness* $\theta(G)$ is the minimum number of planar subgraphs of G whose union is G. It originates in a research problem posed by Harary [30], where he asked for the thickness of K_9. Formulas for the thickness of complete and complete bipartite graphs are given in [6,5,1]; a survey on the thickness can be found in [42]. On the other hand, the *coarseness* $\xi(G)$ is the maximum number of edge-disjoint non-planar subgraphs of G. According to Harary [31], the problem goes back to Erdös who made a slip of the tongue when explaining the concept of thickness. Nevertheless, Beineke, Guy, and Chartrand started to study the problem and developed formulas and bounds for complete and complete bipartite graphs [28,3,29,4].

It is easy to see that the various non-planarity measures are related in the following way:

$$\xi(G) \leq \mu(G) \leq \nu(G)$$
$$\theta(G) - 1 \leq \mu(G) \leq \nu(G).$$

However, computing the crossing number [21], skewness [40], and thickness [41] yield NP-hard optimization problems. For the coarseness, the complexity status is still open.

In this paper, we focus on preprocessing strategies which try to reduce the instance size before solving computationally hard problems. Preprocessing has already been successfully used for other optimization problems, e.g., in the context of railway data [27] or for the prize-collecting Steiner Tree problem [45]. It is well known that, for each of the non-planarity measures, it is sufficient to consider each block of the graph separately. Farr and Eades [20] have shown how the computation of the skewness can be composed of the skewnesses of smaller graphs if the graph has a cut set of at most 4 edges. In contrast to this approach, we consider vertex connectivity. In particular, we exploit the 3-connectivity structure of a 2-connected graph G and derive a graph that behaves invariant to the above non-planarity measures. We call this graph the *non-planar core* of G and give an efficient algorithm for its construction. In order to compute the crossing number, skewness, coarseness, or thickness of G, any standard algorithm can be applied to the non-planar core. This approach targets in particular exact algorithms, since their running times heavily depend on the instance size, but heuristic approaches also benefit from the preprocessing. It is also constructive in the sense that a solution for the non-planar core (e.g., a crossing minimal drawing) implies a solution for G in a straight-forward way.

Various heuristic and exact methods for computing non-planarity measures have been proposed. The standard heuristic for crossing minimization is the planarization approach suggested by Batini, Talamo, and Tamassia [2]. An experimental study by Gutwenger and Mutzel [24] investigates various variations of this approach. Recently, the first exact algorithm for the crossing number problem has been proposed by Buchheim, Chimani et al. [9,8,13] using branch-and-cut techniques. There is a large number of heuristics for finding planar subgraphs [35,19,10,22,18,34]. Moreover, there is an exact branch-and-cut algorithm by Jünger and Mutzel [36], as well as a 4/9 approximation algorithm by Călinescu [11]. The standard approach for computing the thickness consists of iteratively extracting planar subgraphs; see [42]. There is no heuristic published for the coarseness, but an obvious approach is to iteratively extract a Kuratowski subdivision; each such extraction can be done in linear time [14,7].

This paper is organized as follows. After introducing some basic terminology in Section 2, the definition of the non-planar core and a linear-time construction algorithm are presented in Section 3. Section 4 applies the new reduction technique to crossing number, skewness, coarseness, and thickness. Further reductions are considered in Section 5. We show that a straight-forward idea to further reduce the size of the core is not correct by giving counter-examples for crossing number, skewness, and coarseness; on the other hand, we present a much better reduction for the thickness. We conclude the paper with experimental results on the size of the core graph and its improved counterpart for the thickness problem, based on a popular set of benchmark graphs.