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λ-backbone colorings along pairwise disjoint stars and matchings

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ABSTRACT

Given an integer $\lambda \geq 2$, a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring of (G, H) is a proper vertex coloring $V \to \{1, 2, \dots\}$ of G, in which the colors assigned to adjacent vertices in H differ by at least λ. We study the case where the backbone is either a collection of pairwise disjoint stars or a matching. We show that for a star backbone S of G the minimum number ℓ for which a λ-backbone coloring of (G, S) with colors in $\{1, \dots, \ell\}$ exists can roughly differ by a multiplicative factor of at most $2 - \frac{1}{\lambda}$ from the chromatic number $\chi(G)$. For the special case of matching backbones this factor is roughly $2 - \frac{2}{\lambda+1}$. We also show that the computational complexity of the problem "Given a graph G with a star backbone S, and an integer ℓ, is there a λ-backbone coloring of (G, S) with colors in $\{1, \dots, \ell\}$?" jumps from polynomially solvable to NP-complete between $\ell = \lambda + 1$ and $\ell = \lambda + 2$ (the case $\ell = \lambda + 2$ is even NP-complete for matchings). We finish the paper by discussing some open problems regarding planar graphs.

1. Introduction

In [7] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment.

Graphs are used to model the topology and interference between transmitters (receivers, base stations, sensors): the vertices represent the transmitters; two vertices are adjacent if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem is to assign the frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (see, e.g., [16,20]). Although new technologies have led to different ways of avoiding interference between powerful transmitters, such as base stations for mobile telephones, the above coloring problems still apply to less powerful transmitters, such as those appearing in sensor networks.

We refer the reader to [6,7] for an overview of related research, but we repeat the general framework and some of the related research here for convenience and background.

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Given two graphs G_1 and G_2 with the property that G_1 is a spanning subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2.

Many known coloring problems fit into this general framework. We mention some of them here explicitly, without giving details. First of all suppose that $G_2 = G_1^2$, i.e. G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the *distance-2-coloring problem*. Much of the research has been concentrated on the case that G_1 is a planar graph. We refer to [1,4,5,18,21] for more details. In some versions of this problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. A closely related variant is known as the radio coloring problem and has been studied (under various names) in [2,9–13,19]. A third variant is known as the radio labeling problem and models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. We refer to [15,17] for more particulars.

In [7], a situation is modeled in which the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means more restrictions are put on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters.

Postponing the relevant definitions, we consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that models the backbone) differ by at least $\lambda \geq 2$. This is a continuation of the study in [7]. Throughout the paper we consider two types of backbones: matchings and disjoint unions of stars.

Matching backbones reflect the necessity to assign considerably different frequencies to pairwise very close (or most likely interfering) transmitters. This occurs in real world applications such as military scenarios, where soldiers or military vehicles carry two (or sometimes more) radios for reliable communication. Future applications include the use of sensors or sensor tags in clothes or on bodies.

For star backbones one could think of applications to sensor networks. If sensors have low battery capacities, the tasks of transmitting data are often assigned to specific sensors, called cluster heads, that represent pairwise disjoint clusters of sensors. Within the clusters there should be a considerable difference between the frequencies assigned to the cluster head and to the other sensors within the same cluster, whereas the differences between the frequencies assigned to the other sensors within the cluster, or between different clusters, are of secondary importance. This situation is well reflected by a backbone consisting of disjoint stars.

We refer the reader to [7,6] for a more extensive overview of related research, but we repeat the relevant definitions in the next section.

2. Terminology

For undefined terminology we refer to [3].

Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \to \{1, 2, 3, \ldots\}$ is a *vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*. We say that $f(u)$ is the *color* of u. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$ is *independent* if G does not contain edges with both end vertices in V'. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 1$, a vertex coloring f is a *λ-backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. A λ-backbone coloring $f : V \to \{1, \ldots, \ell\}$ is called a *λ-backbone ℓ-coloring*. The *λ-backbone coloring number* $\mathrm{BBC}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone ℓ-coloring. Since a 1-backbone coloring is equivalent to a vertex coloring, we assume from now on that $\lambda \geq 2$. Throughout the manuscript we will reserve the symbol "ℓ" for λ-backbone ℓ-colorings and the symbol "k" for k-colorings.

A *path* is a graph P whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_P = \{v_1v_2, \ldots, v_{n-1}v_n\}$. A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a path connecting u and v. The *length* of a path is the number of its edges. If a graph G contains a spanning subgraph H that is a path, then H is called a *Hamiltonian path*.

A *cycle* is a graph C whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. A *tree* is a connected graph that does not contain any cycles.

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. A graph is called *bipartite* if its vertices can be partitioned into two sets A and B such that each edge has one of its