Discrete Mathematics 309 (2009) 5757–5765

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The circular chromatic number of hypergraphs

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ARTICLE INFO

Article history:
Received 31 August 2006
Accepted 28 May 2008
Available online 26 July 2008

This paper is dedicated to Pavol Hell on the occasion of his sixtieth birthday. The authors are two of his former students, and their former student.

Keywords:
Circular colourings
Hypergraphs

ABSTRACT

A generalization of the circular chromatic number to hypergraphs is discussed. In particular, it is indicated how the basic theory, and five equivalent formulations of the circular chromatic number of graphs, can be carried over to hypergraphs with essentially the same proofs.

1. Introduction

This paper is concerned with extending to hypergraphs the concepts of circular colouring and circular chromatic number of graphs. Starting with a definition of a (k, d)-colouring of a hypergraph due to Eslahchi and Rafiey [6], a combinatorial framework for such colourings in terms of hypergraph homomorphisms is described. This framework mirrors that of Bondy and Hell [5] for circular colourings of graphs. The same results, often in the context of uniform hypergraphs and derived using a slightly different definition of homomorphism, have been noted by Eslahchi and Rafiey [6]. After describing the framework, it is shown how four other formulations of the circular chromatic number of graphs, due respectively to Vince [14], Zhu [18], Barbosa and Gafni [2] (also see the paper by Yeh and Zhu [16]), and Goddyn, Tarsi and Zhang [8] can be extended to hypergraphs. The modifications of the definitions involved are in the same spirit in each case, and the statements and proofs are substantially the same as for graphs. We take this as evidence that the definition and framework used is "the correct" extension of this concept to hypergraphs.

This paper could be considered as a sort of survey on definitions and basic properties of circular colourings. Two comprehensive surveys on the circular chromatic number of graphs have been written by Zhu [17,19]. Circular colourings of graphs, including most of the results mentioned above, are also treated in the book by Hell and Nešetřil [10].

Previous work on such a generalization is due to Haddad and Zhou [9]. They considered two concepts: strong and weak circular colourings of uniform hypergraphs. In a weak (k, d)-circular colouring of a hypergraph H, there is no monochromatic edge and any two adjacent vertices that are not assigned the same colour must have colours that differ by at least d with respect to the circular norm (which is defined below). In a strong (k, d)-circular colouring any two adjacent vertices must have colours that differ by at least d with respect to the circular norm.

The relationship between (k, d)-colourings of uniform hypergraphs and complete uniform subhypergraphs has been studied by Eslahchi and Rafiey [7]. Fractional colourings of hypergraphs are studied by the same authors in the same paper.

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doi:10.1016/j.disc.2008.05.038

Almost all of the results in this paper have been proved elsewhere for graphs. Since the proofs of the corresponding results for hypergraphs are essentially the same – if not exactly the same – each result should be attributed to its originator. Instead of repeating the same citation many times, we cite the original paper at the start of each section, and point out any variations in the statements or proofs. The M.Sc. Thesis of Laura Shepherd [13] is an extended version of this manuscript that includes all of the proofs.

2. Basic definitions

A basic familiarity with graphs, graph colourings and homomorphisms, and hypergraphs is assumed. An excellent reference on graphs, colourings and homomorphisms is the recent book by Hell and Nešetřil [10]. Hypergraphs are thoroughly treated in the book by Berge [3] and the handbook edited by Rosen [12]. A comprehensive reference on graph theory is the text by West [15].

We begin by noting an important subtlety. In this paper, if $H = (V, E)$ is a hypergraph and $X \subseteq V$, then the subhypergraph of H induced by X is the hypergraph with vertex set X and edge set $\{e \in E : e \subseteq X\}$. That is, the edges of an induced subhypergraph of H are the edges of H that are contained in the set X. They are *not* the intersections of edges of H with X. Similarly, the subhypergraph induced by a subset $Y \subseteq E$ has edge set Y and vertex set the union of all edges belonging to Y.

A (u, v)-*walk* in a hypergraph H is a finite, alternating sequence $u = v_0, e_1, v_1, \dots, e_k, v_k = v$ of vertices and edges, in which $v_{i-1}, v_i \in e_i$ for $i = 1, 2, \dots, k$. The vertex u is the *origin* of the walk, and the vertex v is its *terminus*. The integer k is called the *length* of the walk. A walk in which no vertex is repeated in the sequence (but edges can be repeated) is called a *weak path*. A walk is called *closed* if its origin is the same as its terminus. A closed walk of length at least two in which the vertices are distinct except for the origin and terminus and for which edges may be repeated is called a *weak cycle*.

If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are hypergraphs, then a *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that $f(e) \in E(H)$ for every $e \in E(G)$. A homomorphism of G to H will sometimes be denoted by $G \to H$. It is easy to see that the composition of homomorphisms $G \to H$ and $H \to K$ is a homomorphism $G \to K$.

For a positive integer k, a k-*colouring* of a hypergraph $H = (V, E)$ is a function $c : V \to \mathbb{Z}_k$ such that no edge of H is *monochromatic*, i.e. has the same colour assigned to all of its vertices. Equivalently, a k-colouring of a hypergraph H is a homomorphism of H to a complete hypergraph on k vertices. The *chromatic number* of H, denoted $\chi(H)$, is the smallest positive integer k for which there exists a k-colouring of H.

Let $k \in \mathbb{R}^+$ and $x \in [0, k)$. The *circular k-norm* of x is the quantity $|x|_k = \min\{x, k - x\}$. If $[0, k)$ is regarded as the set of points on a circle of circumference k, this is the distance from x to 0 by travelling around the circle.

3. (k, d)-colourings of hypergraphs

In this section the definitions of a (k, d)-colouring and circular chromatic number of a hypergraph are stated, and equivalent formulations in terms of hypergraph homomorphisms is described. For 2-uniform hypergraphs (i.e. graphs), these concepts coincide with their graph counterparts. The results in this section mirror those of Bondy and Hell [5]. The same results, often in the context of uniform hypergraphs and derived using the slightly different definition of homomorphism described below, have been noted by Eslahchi and Rafiey [6]. These results are included so that the paper contains a comprehensive treatment of circular colourings of hypergraphs.

If k and d are positive integers with $\frac{k}{d} \geq 2$, then a (k, d)-*colouring* of a hypergraph $G = (V, E)$ is a function $c : V \to \mathbb{Z}_k$ such that for every edge $e \in E$ there exist vertices x and y in e with $|c(x) - c(y)|_k \geq d$. A *circular colouring* of G is a (k, d)-colouring of G for some pair of integers k and $d \frac{k}{d} \geq 2$. The *circular chromatic number of* H, denoted $\chi_c(H)$, is the infimum of all fractions $\frac{k}{d}$ for which there exists a (k, d)-colouring of H. Examples of circular colourings of hypergraphs are given in a subsequent section, after some theory has been developed.

We now define a family of hypergraphs that facilitate expressing (k, d)-colourings in the language of homomorphisms. If k and d are positive integers with $\frac{k}{d} \geq 2$, then the *circular clique* H_d^k is defined to have vertex-set $\mathbb{Z}_k$, and edge-set the set of all subsets $e \subseteq \mathbb{Z}_k$ with size at least two such that there exist vertices $x, y \in e$ with $|x - y|_k \geq d$. The hypergraphs H_d^k play the same role as do the *circular cliques* (or *fractional complete graphs*) G_d^k (with vertex set $\mathbb{Z}_k$ and edge set $\{xy : |x - y|_k \geq d\}$) in the theory of circular colourings of graphs. In fact, the edges of size two in H_d^k induce a subhypergraph isomorphic to G_d^k.

Proposition 3.1. *There exists a (k, d)-colouring of the hypergraph G if and only if there exists a homomorphism $G \to H_d^k$.*

Thus, for any hypergraph H, the circular chromatic number $\chi_c(H)$ is the infimum of the set of ratios k/d such that $H \to H_d^k$. It follows from the results below that the infimum can be replaced by a minimum, as it can for graphs.

Proposition 3.2. *There exists a k-colouring of the hypergraph G if and only if there exists a homomorphism $G \to H_1^k$.*