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Hall number for list colorings of graphs: Extremal results

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ABSTRACT

Given a graph $G = (V, E)$ and sets $L(v)$ of allowed colors for each $v \in V$, a *list coloring* of G is an assignment of colors $\varphi(v)$ to the vertices, such that $\varphi(v) \in L(v)$ for all $v \in V$ and $\varphi(u) \neq \varphi(v)$ for all $uv \in E$. The *choice number* of G is the smallest natural number k admitting a list coloring for G whenever $|L(v)| \geq k$ holds for every vertex v. This concept has an interesting variant, called *Hall number*, where an obvious necessary condition for colorability is put as a restriction on the lists $L(v)$. (On complete graphs, this condition is equivalent to the well-known one in Hall's Marriage Theorem.) We prove that vertex deletion or edge insertion in a graph of order $n > 3$ may make the Hall number decrease by as much as $n - 3$. This estimate is tight for all n. Tightness is deduced from the upper bound that every graph of order n has Hall number at most $n - 2$. We also characterize the cases of equality; for $n \geq 6$ these are precisely the graphs whose complements are $K_2 \cup (n-2)K_1$, $P_4 \cup (n-4)K_1$, and $C_5 \cup (n-5)K_1$. Our results completely solve a problem raised by Hilton, Johnson and Wantland [A.J.W. Hilton, P.D. Johnson, Jr., E. B. Wantland, The Hall number of a simple graph, Congr. Numer. 121 (1996), 161–182, Problem 7] in terms of the number of vertices, and strongly improve some estimates due to Hilton and Johnson [A.J.W. Hilton, P.D. Johnson, Jr., The Hall number, the Hall index, and the total Hall number of a graph, Discrete Appl. Math. 94 (1999), 227–245] as a function of maximum degree.

1. Introduction

Let $G = (V, E)$ be a finite simple graph, with vertex set $V = \{v_1, \dots, v_n\}$ $(n \geq 1)$ and edge set E. Assume further that a collection $\mathcal{L} = \{L(v_1), \dots, L(v_n)\}$ of finite sets is given; here each 'list' $L(v_i)$ will be viewed as the set of 'allowed colors' for vertex v_i. We shall assume, without loss of generality, that every list $L(v_i)$ is a subset of $\mathbb{N}$, the set of natural numbers. If $|L(v_i)| \geq k$ holds for all lists in $\mathcal{L}$, where k is a positive integer, then we call $\mathcal{L}$ a *k-assignment* on G.

Graph G with a given $\mathcal{L}$ is *list colorable*, or *$\mathcal{L}$-colorable*, if there exists a vertex coloring $\varphi : V \to \mathbb{N}$ such that $\varphi(v_i) \in L(v_i)$ for every i and $\varphi(v_i) \neq \varphi(v_j)$ for every pair $v_iv_j \in E$ of adjacent vertices. That is, φ has to be a proper vertex coloring that respects the list assignment $\mathcal{L}$. The *choice number*, denoted by $\chi_\ell(G)$, is the smallest integer k such that G is $\mathcal{L}$-colorable for every k-assignment $\mathcal{L}$. It was known already from the very first papers [1,15] on the subject that the choice number can be much larger than the chromatic number (there exist bipartite graphs with arbitrarily large χ_ℓ). An overview of results and many references can be found in the subsequent surveys [12,11].

Although the choice number of K_n (the complete graph of order n) is equal to the number n of vertices, the colorability of K_n with lists of any (possibly small and unequal) size is well understood. As a matter of fact, a direct consequence of Hall's Marriage Theorem [6] is that K_n is $\mathcal{L}$-colorable if and only if, for every $k = 1, 2, \dots, n$, the union of any k lists contains at least k colors. This observation leads to the following two interesting notions, introduced by Hilton and Johnson in [7].

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Hall condition. For any vertex subset $X \subseteq V$ and any color c, let $\alpha(X, c)$—or $\alpha(X, c, \mathcal{L})$ if $\mathcal{L}$ has to be emphasized, too—denote the *independence number* of the subgraph of G induced by those vertices $v_i \in X$ whose lists $L(v_i)$ contain c. We say that G satisfies the *Hall Condition* if

$$\sum_{c \in \mathbb{N}} \alpha(X, c) \geq |X| \quad \forall X \subseteq V. \tag{HC}$$

This condition is necessary for $\mathcal{L}$-colorability, but not sufficient in general. The smallest counterexamples are C_4 (the 4-cycle) and $K_4 - e$ (one edge deleted from K_4). An uncolorable $\mathcal{L}$ satisfying (HC) can be created for both of these graphs by assigning the lists $\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ so that the vertices with lists $\{1, 2\}$ and $\{1, 3\}$ are nonadjacent.

Hall number. The *Hall number* of graph G, denoted by $h(G)$, is the smallest positive integer k such that G is $\mathcal{L}$-colorable for every k-assignment satisfying the Hall Condition (HC). In other words, $h(G)$ is the smallest lower bound $k \in \mathbb{N}$ on the list sizes that makes (HC) sufficient for the list-colorability of G. It follows immediately from the definitions that $h(G) \leq \chi_\ell(G)$ holds for every G.

Vertex deletion cannot make the Hall number increase, but simple examples show that edge deletion may yield a subgraph with a larger Hall number; for instance, $h(K_4) = 1$ while $h(K_4 - e) = 2$. In studying these operations, Hilton, Johnson and Wantland [9] raised the following problems: How much can $h(G)$ decrease if a vertex is deleted, or a new edge is inserted ? (For the latter, originally only the weaker question was raised whether the difference between $h(G)$ and $h(G-e)$ can be greater than one.)

As a partial answer, Hilton and Johnson [8] proved the following lower bounds, in terms of the maximum vertex degree $\Delta(G)$ in G.

- For every integer $k \geq 3$, there exists a graph G with $\Delta(G) \in \{k, k + 1, k + 2\}$ containing a vertex v such that $h(G) - h(G - v) \geq k/3$.
- For every integer $k \geq 3$, there exists a graph G with $\Delta(G) = k + 2$ containing an edge e such that $h(G - e) - h(G) \geq k/5$.

It was left as an open problem to investigate the tightness of these lower bounds.

In this paper, a complete solution of the analogous problem in terms of the *number of vertices* is derived from the following result. It will imply improved estimates for the two problems above, too.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 3$, and let G be a graph on n vertices. Then the following are valid.*

(i) $h(G) \leq n - 2$.
(ii) *If $n \geq 6$, then $h(G) = n - 2$ holds if and only if the complement of G is either K_2 or P_4 or C_5 (plus $n - 2$, $n - 4$, or $n - 5$ isolated vertices).*
(iii) *The equality $h(K_n - e) = n - 2$ remains valid for $n = 3, 4, 5$, too.*

In fact, for $3 \leq n \leq 5$, the family of graphs with n vertices and Hall number $n - 2$ differs from the one described above. It can be listed as follows.

- $n = 3$: all graphs.
- $n = 4$: C_4 and $K_4 - e$.
- $n = 5$: the graphs whose complement is one of the following graphs: K_2, $2K_2$, P_3, P_4, $K_2 \cup P_3$, C_3 (plus 3, 1, 2, 1, 0, or 2 isolated vertices, respectively).

Recalling that all complete graphs have Hall number 1, and observing that the deletion of a vertex of degree $n - 2$ from $K_n - e$ yields K_{n-1}, the following tight upper bound is obtained:

Corollary 1. *If G has $n \geq 4$ vertices, then deleting an edge or a vertex from G can increase or decrease, respectively, $h(G)$ by at most $n - 3$. This upper bound is tight for both operations, as shown by the sequence*

$$K_n \to K_n - e \to K_{n-1}.$$

It also turns out (see Section 5) that the sequence exhibited above is the only one verifying the tightness of the upper bound $n - 3$ if $n \geq 6$. Moreover, as both K_n and $K_n - e$ have maximum degree $n - 1$ for $n \geq 3$, we obtain the following improvements in Hilton and Johnson's estimates, showing that the corresponding values $k/3$ and $k/5$ can be replaced with $k - 2$.

Corollary 2. *Let $k \geq 3$ be any integer.*

(i) *There exists a graph G with $\Delta(G) = k$ and containing a vertex v such that $h(G) - h(G - v) = k - 2$.*
(ii) *There exists a graph G with $\Delta(G) = k$ and containing an edge e such that $h(G - e) - h(G) = k - 2$.*

Theorem 1 characterizes the graphs with largest Hall number. From the other end, it was proved by Hilton and Johnson [7] and independently by Gröflin [5] that a graph G has $h(G) = 1$ if and only if every block of G is a complete graph. The case of $h(G) = 2$ is more complicated; its characterization was completed in [3], using several results from the earlier papers [2,10], too.

Related preliminary results, involving parts (i) and (iii) of Theorem 1 and Corollaries 1 and 2, were included in our unpublished manuscript [13] and in the recent electronic publication [14]. Those works contain some ingredients of the present proof techniques, too.