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Enumeration of unrooted hypermaps of a given genus

Alexander Mednykh [a], Roman Nedela [b,*]

[a] Sobolev Institute of Mathematics, Novosibirsk State University, 630090 Novosibirsk, Russia

[b] Institute of Mathematics, Slovak Academy of Sciences, Severná 5, 975 49 Banská Bystrica, Slovakia

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ABSTRACT

In this paper we derive an enumeration formula for the number of hypermaps of a given genus g and given number of darts n in terms of the numbers of rooted hypermaps of genus $\gamma \leq g$ with m darts, where $m|n$. Explicit expressions for the number of rooted hypermaps of genus g with n darts were derived by Walsh [T.R.S. Walsh, Hypermaps versus bipartite maps, J. Combin. Theory B 18 (2) (1975) 155–163] for $g = 0$, and by Arquès [D. Arquès, Hypercartes pointées sur le tore: Décompositions et dénombrements, J. Combin. Theory B 43 (1987) 275–286] for $g = 1$. We apply our general counting formula to derive explicit expressions for the number of unrooted spherical hypermaps and for the number of unrooted toroidal hypermaps with given number of darts. We note that in this paper isomorphism classes of hypermaps of genus $g \geq 0$ are distinguished up to the action of orientation-preserving hypermap isomorphisms. The enumeration results can be expressed in terms of Fuchsian groups.

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1. Introduction

An *oriented map* is a 2-cell decomposition of a closed orientable surface with a fixed global orientation. Generally, maps can be described combinatorially via graph embeddings. Oriented hypermaps are generalisations of oriented maps. While maps are 2-cell embeddings of graphs, hypermaps can be viewed as embeddings of hypergraphs in closed orientable surfaces. Such a model was investigated by Walsh in [29], where the underlying hypergraph is described via the corresponding 2-coloured bipartite graph B, and the hypermap itself is determined by a 2-cell embedding $B \to S$.

By a *map* we mean a 2-cell decomposition of a compact connected surface. Enumeration of maps on surfaces has attracted a lot of attention during the past decades [20]. Generally, problems of the following sort are considered:

Problem 1. How many isomorphism classes of maps with a given property $\mathcal{P}$ and a given number of edges (vertices, faces) are there?

The beginnings of the enumerative theory of maps are closely related to the enumeration of plane trees considered in 1960s by Tutte [26] and Harary, Prins and Tutte [7] (see [8,21] as well). Later many other distinguished classes of maps including triangulations, outerplanar, cubic, Eulerian, nonseparable, simple, loopless, two-face maps etc. were considered. Although there are more than 100 published papers on map enumeration most of them deal with the enumeration of rooted maps with a given property. In particular, there is a lack of results on enumeration of unrooted maps of genus ≥ 1. Most of the results on map enumeration in the unrooted case are restricted to planar maps [14,15,30,31,16]. A recent paper [23] presents a breakthrough in the enumeration problem for unrooted maps of genus ≥ 1. In the present paper we apply the methods employed in [23] to solve an analogous problem for hypermaps.

* Corresponding author.
E-mail addresses: mednykh@math.nsc.ru (A. Mednykh), nedela@savbb.sk (R. Nedela).

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The problem considered in this paper reads as follows.

Problem 2. What is the number $H_g(n)$ of isomorphism classes of oriented unrooted hypermaps of given genus g and given number of darts n?

An oriented map is called *rooted* if one of the darts (arcs) is distinguished as a root. By a *dart* of a map we mean an edge endowed with one of the two possible orientations. Isomorphisms between oriented rooted maps take root onto root. A rooted variant of Problem 2 follows.

Problem 3. What is the number $h_g(n)$ of isomorphism classes of oriented rooted hypermaps of given genus g and given number of darts n?

Problem 3 was solved by Walsh in 1975 [29] for $g = 0$, i.e. for the spherical case. A corresponding case of Problem 2 for $g = 0$, was settled by Bousquet-Mélou and Schaeffer in terms of planar 2-constellations in 2000 [3]. As concerns genera $g \geq 1$, the solution of Problem 3 was obtained by Arqués for $g = 1$ [2], the other instances of Problem 3 remain unsettled.

The aim of the present paper is to show that Problem 2 can be reduced to Problem 3. More precisely, we prove that the numbers of unrooted oriented hypermaps with n darts and given genus g can be determined explicitly whenever the numbers $h_\gamma(m)$ are known for each $m|n$ and $\gamma \leq g$ (see Theorem 3.5 for details). Since $h_\gamma(m)$ are known for $\gamma = 0, 1$ we are able to determine the numbers $H_0(n)$ and $H_1(n)$ in terms of arithmetic functions depending on n. All the derived results can be expressed in group-theoretical language. Namely, the numbers $h_g(n)$ determine the numbers of subgroups of index n and genus g of a free group of rank two, seen as the universal triangle group $\Delta^+(\infty, \infty, \infty) = \langle x, y, z | xyz = 1\rangle$; while $H_g(n)$ give the numbers of conjugacy classes of such subgroups. Note that conjugacy classes of subgroups of free groups of given index were enumerated by Liskovets [17] (see [13,25] as well).

2. Hypermaps on orbifolds

Hypermaps on surfaces. An *oriented combinatorial hypermap* is a triple $\mathcal{H} = (D; R, L)$, where D is a finite set of darts (called brins, blades, bits as well) and R, L are permutations of D such that $\langle R, L\rangle$ is transitive on D. The orbits of R are called *hypervertices*, the orbits of L are called *hyperedges* and the orbits of RL are called *hyperfaces*. The degree of a hypervertex (hyperedge, hyperface) is the size of the respective orbit.

Let $|D| = n$. Denote by v, e and f the numbers of hypervertices, hyperedges and hyperfaces. Then the genus g of $\mathcal{H}$ is given by the Euler–Poincare formula as follows

$$v + e + f - n = 2 - 2g.$$

Given hypermaps $\mathcal{H}_i = (D_i; R_i, L_i)$, $i = 1, 2$ a mapping $\psi : D_1 \to D_2$ such that $R_2\psi = \psi R_1$ and $L_2\psi = \psi L_1$ is called a *morphism* (or a *covering*) $\mathcal{H}_1 \to \mathcal{H}_2$. Note that each morphism between hypermaps is by definition an epimorphism. If $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bijection, ψ is an *isomorphism*. The isomorphisms $\mathcal{H} \to \mathcal{H}$ form a group $\mathrm{Aut}(\mathcal{H})$ of *automorphisms* of $\mathcal{H}$. It is easily seen that $\mathrm{Aut}(\mathcal{H})$ acts semiregularly on D; equivalently, the stabiliser of a dart is trivial. A hypermap $\mathcal{H}$ is called *rooted* if one element x of D is chosen to be a root. Morphisms between rooted hypermaps take roots onto roots. It follows that a rooted hypermap admits no non-trivial automorphisms.

By *a surface* we mean a connected, orientable surface without boundary. A *topological map* is a 2-cell decomposition of a surface. Usually, maps on surfaces are described as 2-cell embeddings of graphs. Oriented combinatorial maps are hypermaps $(D; R, L)$ such that L is a fixed-point-free involution. Walsh observed that oriented hypermaps can be viewed as particular maps. Namely, he proved a one-to-one correspondence [29, Lemma 1] between hypermaps and the set of (oriented) 2-coloured bipartite maps. That means that one of the two global orientations of the underlying surface is fixed, and moreover, we assume that a colouring of vertices, say by black and white colours, is preserved by morphisms between maps. The correspondence is given as follows. Let $\mathcal{M}$ be 2-coloured bipartite map on an orientable surface S with a fixed global orientation. We set D to be the set of edges of $\mathcal{M}$. The orientation of S induces at each black vertex v of $\mathcal{M}$ a cyclic permutation R_v of edges incident with v. This way a permutation $R = \prod R_v$ of D is defined. Similarly, the orientation of S determines at each white vertex u a cyclic permutation L_u. Set $L = \prod L_u$. Hence we have a unique hypermap $(D; R, L)$ corresponding to $\mathcal{M}$. Conversely, given hypermap $(D; R, L)$ we first define a bipartite 2-coloured graph X whose edges are elements of D, black vertices are orbits of R and white vertices are orbits of L. An edge $x \in D$ is incident to a (black or white) vertex u if $x \in u$. The permutations R and L induce local rotations of arcs outgoing from black and white vertices, respectively. It is well known (see Gross and Tucker [6, Section 3.2]) that the system of rotations determines a 2-cell embedding of X into an orientable surface.

Similarly as above, an oriented 2-coloured bipartite map is called *rooted* if one of the edges is selected to be a root. Morphisms between rooted 2-coloured bipartite maps take a root onto a root.

There is yet another way to describe hypermaps. Let $\mathcal{H} = (D; R, L)$ be a hypermap. Clearly, the permutation group $\langle R, L\rangle$ is an epimorphic image of the free product $\Delta^+ = C * C \cong \langle\rho\rangle * \langle\lambda\rangle$ of two infinite cyclic groups. The group Δ^+ acts on D via the epimorphism taking $\rho \mapsto R$ and $\lambda \mapsto L$. Thus by using some standard results in permutation group theory each hypermap can be described by a subgroup $F \leq \Delta^+$ [11,27,28,5]. The subgroup F, called a *hypermap subgroup*, can be identified with a stabiliser of a dart in the action of Δ^+ on D. Since the action of Δ^+ on D is transitive, the number of darts $|D| = n$ coincides with index $[\Delta^+ : F]$ of F in Δ^+. Given $F \leq \Delta^+$ the corresponding hypermap can be constructed as an

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