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Local topology of the free complex of a two-dimensional generalized convex shelling☆

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Abstract

A generalized convex shelling was introduced by Kashiwabara et al. for their representation theorem of convex geometries. Motivated by the work by Edelman and Reiner, we study local topology of the free complex of a two-dimensional separable generalized convex shelling. As a result, we prove a deletion of an element from such a complex is homotopy equivalent to a single point or two distinct points, depending on the dependency of the element to be deleted. Our result resolves an open problem by Edelman and Reiner for this case, and it can be seen as a first step toward the complete resolution from the viewpoint of a representation theorem for convex geometries by Kashiwabara et al.

Keywords: Abstract convex geometry; Discrete geometry; Topological combinatorics

1. Introduction

A convex geometry is introduced by Edelman and Jamison [4] as a combinatorial abstraction of "convexity" appearing in many objects. Recently, a representation theorem for convex geometries has been established by Kashiwabara et al. [8], which states that every convex geometry is isomorphic to some "separable generalized convex shelling." A generalized convex shelling is defined in terms of two finite point sets in a certain dimension. Therefore, their representation theorem gives a stratification of the convex geometries by the minimum dimension in which a convex geometry can be realized as a separable generalized convex shelling. We study the topology of the free complex of a two-dimensional generalized convex shelling. As a result, we prove the following. (The necessary definitions will be given later.)

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Theorem 1. *Let P and Q be nonempty finite point sets in $\mathbb{R}^2$ such that $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$. In addition, let $\mathscr{L}$ be the generalized convex shelling on P with respect to Q. Consider the free complex $\mathrm{Free}(\mathscr{L})$ of $\mathscr{L}$. Then the following holds.*

1. *If $\mathrm{Dep}_{\mathscr{L}}(x) \neq P$, then the deletion $\mathrm{del}_{\mathrm{Free}(\mathscr{L})}(x)$ of an element $x \in P$ is contractible (i.e., homotopy equivalent to a single point).*
2. If $\mathrm{Dep}_{\mathscr{L}}(x) = P$, *then $\mathrm{del}_{\mathrm{Free}(\mathscr{L})}(x)$ is contractible or homotopy equivalent to a zero-dimensional sphere (i.e., two distinct points).*

The motivation of this work stems from Edelman and Reiner [5]. An Euler–Poincaré type formula for the number of interior points in a d-dimensional point configuration was proved by Ahrens et al. [1] for $d = 2$, and proved by Edelman and Reiner [5] and Klain [10] independently for arbitrary d. The approach by Klain [10] used a more general theorem on valuation, while that by Edelman and Reiner [5] was topological. (Another proof based on oriented matroids was given by Edelman et al. [6].) In the paper by Edelman and Reiner [5], they studied the topology of deletions of the free complex of a convex shelling (arising from a point configuration), and also mentioned a possible generalization to a convex geometry. More precisely speaking, their open problems are as follows.

Open Problem 2 (*Edelman and Reiner [5]*). *Let $\mathscr{L}$ be a convex geometry on E and denote the free complex of $\mathscr{L}$ by* $\mathrm{Free}(\mathscr{L})$.

1. *Is the deletion $\mathrm{del}_{\mathrm{Free}(\mathscr{L})}(x)$ of an element $x \in E$ contractible if $\mathrm{Dep}_{\mathscr{L}}(x) \neq E$?*
2. *Is $\mathrm{del}_{\mathrm{Free}(\mathscr{L})}(x)$ homotopy equivalent to a bouquet of spheres if $\mathrm{Dep}_{\mathscr{L}}(x) = E$?*

Edelman and Reiner [5] showed that this generalization can be successfully done for poset double shellings and simplicial shellings of chordal graphs. Subsequently Edelman et al. [6] showed that this can also successfully be done for a convex shelling of an acyclic oriented matroid. Theorem 1 states that this can also be done for a two-dimensional separable generalized convex shelling. However, our case is not just a special case. Thanks to Kashiwabara et al. [8], every convex geometry is isomorphic to some generalized convex shelling. An explicit statement is as follows.

Proposition 3 (*Kashiwabara et al. [8]*). *For every convex geometry $\mathscr{L}$ on a finite set E, there exist a natural number d and two point sets $P, Q \subseteq \mathbb{R}^d$ satisfying $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$ such that $\mathscr{L}$ is isomorphic to the generalized convex shelling of P with respect to Q.*

Therefore, our result is a step toward a resolution of Open Problem 2.

The organization of this paper is as follows. In the next section we introduce the necessary terminology about simplicial complexes and convex geometries. Section 3 sketches the proof of our theorem. We conclude the paper in Section 4 with some examples.

2. Preliminaries

In this article, we assume a moderate familiarity with graph theory.

2.1. Simplicial complexes

Let E be a finite set. An *abstract simplicial complex* on E is a nonempty family Δ of subsets of E satisfying that: if $X \in \Delta$ and $Y \subseteq X$ then $Y \in \Delta$. Often an abstract simplicial complex is simply called a simplicial complex, and in the literature they are also called independence systems and hereditary set systems. For a simplicial complex Δ on E, a subset of E is called a *face* of the simplicial complex Δ if it belongs to Δ; if not it is called a *nonface*.

For a simplicial complex Δ on E and an element $x \in E$, the *deletion* of x in Δ is defined by $\mathrm{del}_{\Delta}(x) := \{X \in \Delta : x \notin X\}$. Note that the deletion is a simplicial complex on $E \setminus \{x\}$.

When we talk about topology of a simplicial complex, we refer to a geometric realization of the simplicial complex. For details, see Matoušek's book [11].