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Finite classical groups and flag-transitive triplanes

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ABSTRACT

Let $\mathcal{D}$ be a finite nontrivial triplane, i.e. a 2-$(v, k, 3)$ symmetric design, with a flag-transitive, point-primitive automorphism group G. If G is almost simple, with the simple socle X of G being a classical group, then $\mathcal{D}$ is either the unique $(11, 6, 3)$-triplane, with $G = PSL_2(11)$ and $G_\alpha = A_5$, or the unique $(45, 12, 3)$-triplane, with $G = X : 2 = PSp_4(3) : 2 \cong PSU_4(2) : 2$ and $G_\alpha = X_\alpha : 2 = (2^{\cdot}(A_4 \times A_4).2) : 2$, where α is a point of $\mathcal{D}$.

1. Introduction

A 2-(v, k, λ) *symmetric design* is an incidence structure $\mathcal{D} = (P, \mathcal{B})$ where P is a set of points and $\mathcal{B}$ is a set of blocks with an incidence relation such that: (i) $|P| = |\mathcal{B}| = v$, (ii) every block is incident with exactly k points, and (iii) every 2-element subset of P is incident with exactly λ blocks. It is also called a λ-*plane*. $\mathcal{D}$ is called *nontrivial* if $\lambda < k < v - 1$, and the *order* of $\mathcal{D}$ is $n = k - \lambda$. An *automorphism* of a design $\mathcal{D}$ is a permutation of the points which also permutes the blocks, preserving the incidence relation. The set of automorphisms of a design with the composition of functions is a group. If G is a primitive permutation group on the point set P then G is called *point-primitive*, otherwise *point-imprimitive*. A *flag* in a symmetric design is a point-block pair, such that the point is incident with the block. Thus to say that G is *flag-transitive* means that if α_1, α_2 are points, B_1, B_2 are blocks, and α_i is incident with B_i for $i = 1, 2$, then there is an automorphism of $\mathcal{D}$ taking (α_1, B_1) to (α_2, B_2).

Symmetric designs with λ small are of interest. For example, those with $\lambda = 1$ are the *projective planes*, while those with $\lambda = 2$ are called *biplanes*. Flag-transitivity is just one of many conditions that can be imposed on the automorphism group G of a symmetric design $\mathcal{D}$. For the flag-transitive projective planes, Kantor [9] proved that either $\mathcal{D}$ is a Desarguesian projective plane and $PSL(3, n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, where n is even and $n^2 + n + 1$ is prime. In [22–25], using the theorem of classification of finite simple groups, Regueiro reduced the classification of flag-transitive biplanes to the situation where the automorphism group is a 1-dimensional affine group.

For the case $\lambda = 3$, a $(v, k, 3)$-symmetric design is called a *triplane*. The parameters of triplanes with $n = k - 3 \le 25$, $k \le v/2$ and $4n - 1 < v < n^2 + n + 1$ satisfying $k(k - 1) = 3(v - 1)$ and the Bruck–Ryser–Chowla condition are listed in [3, p. 80]. In this range, the only known examples of triplanes correspond to $n = 6, 7, 9, 12$ are $(v, k) = (25, 9), (31, 10), (45, 12), (71, 15)$, but for $n = 13, 16, 19, 22, 25$, $(v, k) = (81, 16), (115, 19), (155, 22), (201, 25), (253, 28)$ respectively, the existence is still undecided.

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After the work of Regueiro on biplanes, we propose the following interesting problem:

Problem 1. Can we classify the triplanes with flag-transitive automorphism groups?

In this paper, we discuss this classification problem. The other motivation about Problem 1 comes from the recent paper [15] by M. Law, C. E. Praeger and S. Reichard, which classifies the (96, 20, 4)-symmetric designs admitting a flag-transitive point-imprimitive automorphism groups. In [15], the authors also suggested a similar classification problem.

In [28], Praeger and Zhou discussed the flag-transitive point-imprimitive (v, k, λ)-symmetric designs. From [28, Theorem 1.1 and Table 1 of Corollary 1.3], we know that if a triplane admits a flag-transitive point-imprimitive automorphism group, then it has parameters (45, 12, 3). Recently, C. E. Praeger proved in [26] that there are exactly two nonisomorphic triplanes with parameters (45, 12, 3), one is point-imprimitive, while the other is point-primitive. Therefore, if any other triplane admits a flag-transitive automorphism group G, then it must be point-primitive. It was shown in [22, Theorem 2] using the O'Nan–Scott Theorem [17] that if $\mathcal{D}$ is a (v, k, λ)-symmetric design admitting a flag-transitive, point-primitive automorphism group G with $\lambda \le 3$, then G is of affine, or almost simple type. So we have the following

Proposition 1.1. *If $\mathcal{D}$ is a triplane other than $(45, 12, 3)$ admitting a flag-transitive automorphism group G, then G is point-primitive and G is of affine, or almost simple type.*

Here G is almost simple means that $X \trianglelefteq G \le Aut(X)$ for some nonabelian simple group X, and G is of affine type means that G is isomorphic to the split extension of V by H, where V is a vector space of size p^d for some prime p and some d, and $H \le GL(V) \cong GL_d(p)$ with H irreducible on V, here the number of points in $\mathcal{D}$ is p^d. So for Problem 1 we only need to consider the following cases:

(1) G is of affine type;
(2) X is a sporadic simple group;
(3) X is an alternating group;
(4) X is an exceptional group of Lie type;
(5) X is a classical group.

Recently, it has been shown in [32,33] that X cannot be a sporadic group or an exceptional group of Lie type. Here we deal with case (5) and prove the following:

Theorem 1.2. *If a triplane $\mathcal{D} = (P, \mathcal{B})$ admits a primitive flag-transitive automorphism group G of almost simple type, with the simple socle X of G being a finite classical group, then one of the following holds:*

(i) *$\mathcal{D}$ is the unique $(11, 6, 3)$-triplane, with $G = X = PSL_2(11)$ and the point stabilizer $G_\alpha = A_5$,*
(ii) *$\mathcal{D}$ is the unique $(45, 12, 3)$-triplane, with $G = X{:}2 = PSp_4(3){:}2 \cong PSU_4(2){:}2$ and the point stabilizer $G_\alpha = X_\alpha{:}2 = (2^{\cdot}(A_4 \times A_4).2){:}2$.*

We will prove Theorem 1.2 in Section 3. Our proof will proceed as in [29,24,25], in which the case for finite linear spaces, biplanes with a flag-transitive automorphism group is treated, respectively.

Theorem 1.2, together with [32,33], yields the following:

Corollary 1.3. *If $\mathcal{D}$ is a nontrivial triplane with a flag-transitive automorphism group G, then one of the following holds:*

(i) *$\mathcal{D}$ has parameters $(45, 12, 3)$,*
(ii) *$\mathcal{D}$ has parameters $(11, 6, 3)$,*
(iii) *G is of affine type,*
(iv) *$Soc(G)$ is an alternating group.*

2. Preliminary results

We assume throughout this paper that $q = p^e$ for p a prime and e a positive integer. If n is a positive integer, then n_p denotes the p-part of n and $n_{p'}$ denotes the p'-part of n. In other words, $n_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$, and $n_{p'} = n/n_p$.

Let $\mathcal{D}$ be a finite nontrivial triplane, and G be a flag-transitive, point-primitive automorphism group of $\mathcal{D}$. We shall assume that G has the socle X which is a simple classical group with a natural projective action on a vector space V of dimension n over the field F (note that we take $F = GF(q)$ if X is not a unitary group, and $F = GF(q^2)$ if $X = U_n(q)$), i.e. $X \trianglelefteq G \le Aut(X)$ and X is one of the finite classical groups: $L_n(q)$, $U_n(q)$, $PSp_{2m}(q)$ $(m \ge 2)$, $P\Omega_{2m+1}(q)$ $(m \ge 3, q \text{ odd})$, $P\Omega^+_{2m}(q) (m \ge 4)$ and $P\Omega^-_{2m}(q) (m \ge 4)$. Our aim is to classify all possible pairs $(\mathcal{D}, G)$.

If H is the stabilizer in G of a point α of $\mathcal{D}$, i.e. $H = G_\alpha$, then H is maximal in G, by Aschbacher's Theorem [1], the stabilizer H lies in one of the families $\mathcal{C}_i$ $(1 \le i \le 8)$ of subgroups of $\Gamma L_n(q)$, or in the set $\mathcal{S}$ of almost simple subgroups not contained in any of these families. We will discuss our problem using these families $\mathcal{C}_i \cup \mathcal{S}$ and the classification of primitive permutation groups of odd degree(see Lemma 2.9). In describing the Aschbacher's geometric subgroups, we denote by $\hat{}K$ the pre-image of the group K in the corresponding linear group.