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# Total colorings and list total colorings of planar graphs without intersecting 4-cycles ☆

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ABSTRACT

Suppose that $G$ is a planar graph with maximum degree $\Delta$ and without intersecting 4-cycles, that is, no two cycles of length 4 have a common vertex. Let $\chi''(G)$, $\chi'_l(G)$ and $\chi''_l(G)$ denote the total chromatic number, list edge chromatic number and list total chromatic number of $G$, respectively. In this paper, it is proved that $\chi''(G) = \Delta + 1$ if $\Delta \geq 7$, and $\chi'_l(G) = \Delta$ and $\chi''_l(G) = \Delta + 1$ if $\Delta(G) \geq 8$. Furthermore, if $G$ is a graph embedded in a surface of nonnegative characteristic, then our results also hold.



## 1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [6] for terminology and notation not defined here. Let $G$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $\Delta$ and $\delta$) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to $v$, and let $d(v) = |N(v)|$ denote the degree of $v$. A *$k$-vertex* or a *$k^+$-vertex* is a vertex of degree $k$ or at least $k$, respectively. A *$k$-cycle* is a cycle of length $k$, and a 3-cycle is usually called a *triangle*.

A *total-$k$-coloring* of a graph $G$ is a coloring of $V \cup E$ using $k$ colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ has a total-$k$-coloring. Clearly, $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [20] posed independently the following famous conjecture, which is known as the *Total Coloring Conjecture* (TCC).

**Conjecture A** (*TCC*). *For any graph $G$, $\chi''(G) \leq \Delta + 2$.*

Conjecture A was confirmed for a graph with $\Delta = 3$ by Rosenfeld [15] and Vijayaditya [19] and for $\Delta \leq 5$ by Kostochka [14]. In recent years, the study of total colorings for the class of planar graphs has attracted considerable attention. For planar graphs Conjecture A was confirmed for $\Delta \geq 7$ [2,13,17]. In 1989, Sánchez-Arroyo [16] proved that deciding whether $\chi''(G) = \Delta + 1$ is NP-complete. But for planar graphs with large maximum degree, it is possible to determine $\chi''(G)$ precisely. It is shown that $\chi''(G) = \Delta + 1$ if $G$ is a planar graph with $\Delta \geq 11$ [4] and $\Delta = 10$ [21]. Borodin et al. [5] also obtained several related results by adding girth restrictions. Wang and Wu [23] considered planar graphs without 4-cycles and got some interesting results. Recently, Sun et al. [18] showed that $\chi''(G) = \Delta + 1$ for planar graphs with $\Delta \geq 9$ and without adjacent triangles.

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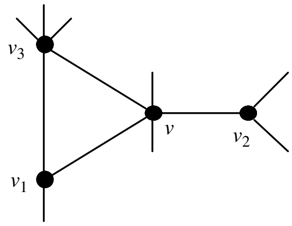


**Fig. 1.** Reducible configuration.

A mapping $L$ is said to be a *total assignment* for a graph $G$ if it assigns a list $L(x)$ of possible colors to each element $x \in V \cup E$. If $G$ has a total coloring $\phi$ such that $\phi(x) \in L(x)$ for all $x \in V \cup E$, and no two adjacent or incident elements receive the same color, then we say that $G$ is *total-L-colorable* or $\phi$ is a *total-L-coloring* of $G$. A graph $G$ is *total-k-choosable* if it is total-$L$-colorable for every total assignment $L$ satisfying $|L(x)| \geq k$ for each $x \in V \cup E$. The *list total chromatic number* $\chi_l''(G)$ of $G$ is the smallest integer $k$ such that $G$ is total-$k$-choosable. The *list edge chromatic number* $\chi_l'(G)$ of $G$ can be defined similarly in terms of coloring the edges alone. The ordinary edge chromatic number of $G$ is denoted by $\chi'(G)$. It follows directly from the definition that $\chi_l'(G) \geq \chi'(G) \geq \Delta$ and $\chi_l''(G) \geq \chi''(G) \geq \Delta + 1$.

List edge and list total colorings are two widely studied generalizations of the classical notions of graph coloring, and quite a few interesting results about these two colorings of planar graphs have been obtained in recent years. Now we introduce a famous conjecture.

**Conjecture B.** *Suppose that $G$ is a graph. Then*

(a) $\chi_l'(G) = \chi'(G)$; (b) $\chi_l''(G) = \chi''(G)$.

Part (a) of Conjecture B was formulated independently by a number of people, including Vizing, Gupta, Albertson and Collins, Bollobás and Harris (see [7] or [13]), and it is well known as the *List Coloring Conjecture*. Part (b) was formulated independently by Borodin et al. [3], Juvan et al. [12] and Hilton et al. [8]. This conjecture has been proved only for a few special cases. Borodin et al. [3] proved $\chi_l'(G) = \chi'(G) = \Delta$ and $\chi_l''(G) = \chi''(G) = \Delta + 1$ for graphs with $\Delta \geq 12$ which can be embedded in a surface of nonnegative characteristic. Wang and Lih [22] proved Conjecture B for outerplanar graphs, and Hou et al. [9] proved it for planar graphs with $\Delta \geq 7$ and without 4-cycles. Some other related results can be seen in [10, 11].

In this paper, we consider planar graphs without intersecting 4-cycles and get the following results.

**Theorem 1.** *Suppose that $G$ is a planar graph without intersecting 4-cycles. If $\Delta \geq 7$, then $\chi''(G) = \Delta + 1$.*

**Theorem 2.** *Suppose that $G$ is a planar graph without intersecting 4-cycles. If $\Delta \geq 8$, then $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$.*

## 2. Proof of Theorem 1

We will introduce some more notation and definitions here for convenience. Let $G = (V, E, F)$ be a planar graph, where $F$ is the face set of $G$. For $f \in F$, we use $b(f)$ to denote the boundary of $f$, and write $f = v_1v_2 \cdots v_n$ if $v_1, v_2, \ldots, v_n$ are the boundary vertices of $f$ in the clockwise order. The *degree* of a face $f$, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A *k-face* or a $k^+$*-face* is a face of degree $k$ or at least $k$, respectively. We use $(d_1, d_2, \ldots, d_n)$ to denote a cycle (or a face) whose boundary vertices are of degree $d_1, d_2, \ldots, d_n$ in the clockwise order in $G$. Let $\delta(f)$ denote the minimum degree of vertices incident with $f$. We say that two cycles (or faces) are *adjacent* if they share at least one edge. A 2-*alternating cycle* is an even cycle $v_1v_2 \cdots v_{2t}v_1$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$; this key concept was introduced in [2] and used in a number of subsequent papers. For a total coloring $\phi$ of $G$ and a vertex $v \in V(G)$, let $\Phi(v) = \{\phi(uv) \mid u \in N(v)\}$ and $\overline{\Phi}(v) = \Phi(v) \cup \{\phi(v)\}$.

Our proofs of the main results are based on the discharging method. In the beginning we define an initial charge for each element. Then we redistribute the charges so that the new charges are nonnegative and the total charge is still the same as before, which leads to a contradiction to Euler's formula.

**Proof of Theorem 1.** Let $G = (V, E, F)$ be a minimal counterexample to the theorem in terms of the total number of vertices and edges. Then every proper subgraph of $G$ is total-$(\Delta + 1)$-colorable. It is easy to see that $G$ is 2-connected and hence, it has no vertices of degree 1 and the boundary $b(f)$ of each face $f$ in $G$ is exactly a cycle (i.e., $b(f)$ cannot pass through a vertex $v$ more than once). Furthermore, $G$ has the following properties.

(i) $G$ contains no edge $uv$ with $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta}{2} \rfloor$ and $d(u) + d(v) \leq \Delta + 1$;
(ii) $G$ contains no 2-alternating cycle;
(iii) $G$ contains no subgraph isomorphic to the configuration in Fig. 1 such that $d(v_1) = d(v_2) = \Delta + 2 - d(v)$ and $\Delta - 1 \leq d(v) \leq \Delta$;
(iv) $G$ contains no 3-face incident with more than one 4-vertex.